Example of Linear Programming Problem Using Matlab to Perform Matrix Calculations

Problem to be Solved

$$\text{Maximize } Z = 4x_1 + 9x_2$$

Subject to:

$$x_2 \le 19$$

$$1.45x_1 + x_2 \le 45$$

$$x_2 - 1.25x_1 \le 10$$

and

$$x_1 \ge 0,\ \ x_2 \ge 0$$

Excel Solver Solution

$$\text{Maximize } Z = 4x_1 + 9x_2$$

Subject to:

$$x_2 \le 19$$

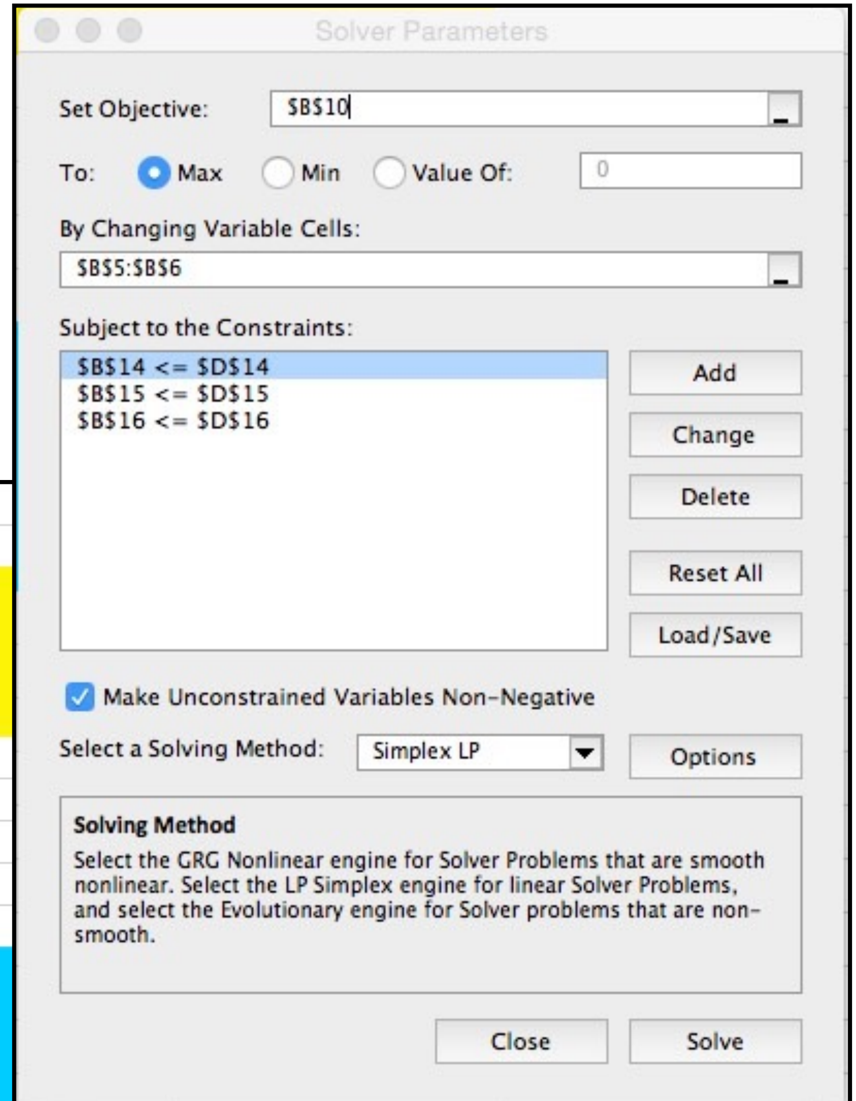
$$1.45x_1 + x_2 \le 45$$

$$x_2 - 1.25x_1 \le 10$$

and

$$x_1 \ge 0,\ x_2 \ge 0$$

Solver Panel



Excel Setup

Optimization Problem - Problem 3 in Q1 - 2015			
Decision Variables			
x1	17.93		
x2	19.00		
Objective Function			
4 x1 + 9 x2	242.72		
Constraint Equations			
	Formula		
x2 <= 19	19.00	<=	
1.45 x1 + x2 <= 45	45.00	<=	
x2-1.25x1<=10	-3.41	<=	10

Converting Inequality Constraints in LP Problems to Standard Form

Type of Constraint	**How to handle**
$3x_1 + 2x_2 \le 180$	Add a slack variable
$3x_1 + 2x_2 = 180$	Add an artificial variable Add a penalty to OF (BigM)
$3x_1 + 2x_2 \ge 180$	Add a negative slack and a positive artificial variable

Excel Solver Solution

Original Problem

$$\text{Maximize } Z = 4x_1 + 9x_2$$

Subject to:

$$x_2 \le 19$$

$$1.45x_1 + x_2 \le 45$$

$$x_2 - 1.25x_1 \le 10$$

and

$$x_1 \ge 0,\ x_2 \ge 0$$

Conversion to Standard Form

$$\text{Maximize } Z = 4x_1 + 9x_2$$

Subject to:

$$x_2 + x_3 = 19$$

$$1.45x_1 + x_2 + x_4 = 45$$

$$x_2 - 1.25x_1 + x_5 = 10$$

and

$$x_1 \ge 0,\ x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0$$

For each inequality constraint of type $\le$ we have added a slack variable

Initial Table (to get an Initial Basic Feasible Solution)

$$Z - 4x_1 - 9x_2 + 0x_3 + 0x_4 + 0x_5 = 0$$

$$x_2 + x_3 = 19$$

$$1.45x_1 + x_2 + x_4 = 45$$

$$x_2 - 1.25x_1 + x_5 = 10$$

In Z-row bring the right hand side (RHS) terms to the left hand side of the equation

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-4	-9	0	0	0	0
x_3	0	0	1	1	0	0	19
x_4	0	1.45	1	0	1	0	45
x_5	0	-1.25	1	0	0	1	10

Initial Basic Feasible Solution (IBFS) is: $x_1 = 0$, $x_2 = 0$, $x_3 = 19$, $x_4 = 45$ and $x_5 = 10$.
Value of the objective function Z = 0.

Iterations (Marching to 2nd Table)

1. Select Pivot column containing Non-Basic variable x_2. The coefficient of x_2 in the Z-row is the most negative and hence improves the solution of Z the most.
2. Take the ratio test. RHS/coefficients in Pivot column.

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS	Ratio test
Z	1	-4	-9	0	0	0	0	
x_3	0	0	1	1	0	0	19	19
x_4	0	1.45	1	0	1	0	45	45
x_5	0	-1.25	1	0	0	1	10	10

Pivot row selected with the minimum ratio of RHS/coefficients in Pivot column

Iterations (Next Steps)

3. Select the lowest ratio (variable x_5 leaves the Basic Variable set and becomes zero in the next table.

4. Variable x_2 enters the solution in the next table.

5. Perform row operations to eliminate all coefficients in Pivot Column (except the intersection of Pivot column and Pivot row)

   - Multiply row with variable x_5 (3rd constraint equation) by 9 and add to Z-row

   - Multiply row with variable x_5 (3rd constraint equation) by (-1) and add to second row (first constraint equation)

6. Eliminate all coefficients in the Pivot column except for the unit value in the Pivot row (see table on next page).

Matlab Matrix Operations

Use Matlab to perform the calculations

Define a matrix **a** that contains all the coefficients in the previous table

```
a = [1 -4 -9     0   0   0   0
      0   0   1   1   0   0   19
      0   1.45 1  0   1   0   45
      0  -1.25 1  0   0   1   10];
```

Define rows r1, r2 , .. r4 as the rows of matrix **a**

```
r1 = a(1,:)
r2 = a(2,:)
r3 = a(3,:)
r4 = a(4,:)
```

a(1,:) means select all elements of row 1

Matlab Matrix Operations (2)

Perform row operations in matrix **a** or now in matrices r1, r2, r3 and r4

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-4	-9	0	0	0	0
x_3	0	0	1	1	0	0	19
x_4	0	1.45	1	0	1	0	45
x_5	0	-1.25	1	0	0	1	10

b = r4*9 + r1

Equivalent to:

b = Pivot row * (9) + Z-row

Yields:

b = [1 -15.25 0 0 0 9 90]

Replace the **Z-row** (matrix **r1**) for matrix **b**

b = r1

Matlab Matrix Operations (2)

Perform row operations in matrix **a** using matrix r2

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-15.25	0	0	0	9	90
x_3	0	0	1	1	0	0	19
x_4	0	1.45	1	0	1	0	45
x_5	0	-1.25	1	0	0	1	10

c = r4*(-1) + r2

Equivalent to:

c = Pivot row * (-1) + row(2)

Yields:

c = [0 2.7 0 0 1 -1 35]

Replace matrix **c** for second row in matrix **a** (or **r2**)

c = r2

Matlab Matrix Operations (2)

Perform row operations in matrix **a** using matrix **r3**

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-15.25	0	0	0	9	90
x_3	0.0	1.25	0.0	1.0	0.0	-1.0	9.0
x_4	0	1.45	1	0	1	0	45
x_5	0	-1.25	1	0	0	1	10

d = r4*(-1) + r3

Equivalent to:

d = Pivot row * (-1) + row(3)

Yields:

d = [0 1.25 0 1 0 -1 9]

Replace matrix **d** for third row in matrix **a** (or r3)

d = r3

Second Table

The new matrices **b**, **c**, and **d** are now substituted back to form a new matrix **a** that is the second table in our problem

a(1,:) = b;
a(2,:) = c;
a(3,:) = d;

The last row in matrix **a** does not to be re-defined since it was the Pivot row and was not modified

Second Table

```
a = [1    -15.25   0   0   0   9   90
     0    2.70  0   0   1   -1  35
     0    1.25  0   1   0   -1  9
     0    -1.25 1   0   0   1   10];
```

New solution is not optimal
Coefficient of x_1 is negative

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1.00	-15.25	0.00	0.00	0.00	9.00	90.00
x_3	0.00	1.25	0.00	1.00	0.00	-1.00	9.00
x_4	0.00	2.70	0.00	0.00	1.00	-1.00	35.00
x_2	0.00	-1.25	1.00	0.00	0.00	1.00	10.00

New solution is: $x_1 = 0$, $\mathbf{x_2 = 10}$, $x_3 = 35$, $x_4 = 9$ and $x_5 = 0$.
Value of the objective function **Z = 90**.

Marching to 3rd Table

1. Select column x_1 as the Pivot column
2. Take ratio test and select second row as the Pivot row
3. Perform row operations to eliminate all coefficients of Pivot column (except the coefficient at the intersection of Pivot row and Pivot column)

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS	Ratio test
Z	1.00	-15.25	0.00	0.00	0.00	9.00	90.00	
x_3	0.00	1.25	0.00	1.00	0.00	-1.00	9.00	7.20
x_4	0.00	2.70	0.00	0.00	1.00	-1.00	35.00	12.96
x_2	0.00	-1.25	1.00	0.00	0.00	1.00	10.00	-8.00

Marching to 3rd Table

1. To facilitate matters start doing row operations on the second row to make coefficient at the intersection of the Pivot row and Pivot column equal to one
2. Divide row (2) by 1.25

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1.00	-15.25	0.00	0.00	0.00	9.00	90.00
x_3	0.00	1.25	0.00	1.00	0.00	-1.00	9.00
x_4	0.00	2.70	0.00	0.00	1.00	-1.00	35.00
x_2	0.00	-1.25	1.00	0.00	0.00	1.00	10.00

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1.00	-15.25	0.00	0.00	0.00	9.00	90.00
x_3	0.00	1.00	0.00	0.80	0.00	-0.80	7.20
x_4	0.00	2.70	0.00	0.00	1.00	-1.00	35.00
x_2	0.00	-1.25	1.00	0.00	0.00	1.00	10.00

Matlab Matrix Operations

Define a matrix **a** that contains all the coefficients in the previous table

```
a = [1    -15.25   0   0   0   9   90
      0    1   0   0.80  0   -0.80 7.20
      0    2.70  0   0   1   -1 35
      0    -1.25 1   0   0   1   10];
```

Define rows r1, r2 , .. r4 as the rows of matrix **a**

```
r1 = a(1,:)
r2 = a(2,:)
r3 = a(3,:)
r4 = a(4,:)
```

Matlab Matrix Operations (2)

Perform row operations in matrix **a** (or r1, r3 and r4)

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1.00	-15.25	0.00	0.00	0.00	9.00	90.00
x_3	0.00	1.00	0.00	0.80	0.00	-0.80	7.20
x_4	0.00	2.70	0.00	0.00	1.00	-1.00	35.00
x_2	0.00	-1.25	1.00	0.00	0.00	1.00	10.00

b = r2*(15.25) + r1

Equivalent to:

b = Pivot row * (15.25) + Z-row

Yields:

b = [1 0   0   12.2   0   -3.2   199.8]

Matrix **b** will replace row (1) in the new matrix **a**

Matlab Matrix Operations (2)

Perform row operations to eliminate coefficient of cell in Pivot column on the third row

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	**1.00**	0.00	0.00	12.20	0.00	-3.20	199.80
x_3	**0.00**	1.00	0.00	0.80	0.00	-0.80	7.20
x_4	**0.00**	2.70	0.00	0.00	1.00	-1.00	35.00
x_2	**0.00**	-1.25	1.00	0.00	0.00	1.00	10.00

c = r2*(-2.7) + r3

Equivalent to:

c = Pivot row * (-2.7) + Z-row

Yields:

c = [0 0   0   -2.16 1   1.16   15.56]

Matrix **c** will replace row (3) in the new matrix **a**

Matlab Matrix Operations (2)

Perform row operations to eliminate coefficient of cell in Pivot column in the fourth row

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	**1.00**	0.00	0.00	12.20	0.00	-3.20	199.80
x_3	**0.00**	1.00	0.00	0.80	0.00	-0.80	7.20
x_4	**0.00**	0.00	0.00	-2.16	1.00	1.16	15.56
x_2	**0.00**	-1.25	1.00	0.00	0.00	1.00	10.00

d = r2*(1.25) + r4

Equivalent to:

d = Pivot row * (-1.25) + Z-row

Yields:

d = [0 0 1 1 0 0 19]

Matrix **d** will replace row (4) in the new matrix **a**

Third Table

The new matrices **b**, **c**, and **d** are now substituted back to form a new matrix **a** that is the second table in our problem

a(1,:) = b;
a(3,:) = c;
a(4,:) = d;

The last row in matrix **a** does not to be re-defined since it was the Pivot row and was not modified

Third Table

a = [1.00 0.00 0.00 12.20 0.00 -3.20 199.80
0.00 1.00 0.00 0.80 0.00 -0.80 7.20
0.00 0.00 0.00 -2.16 1.00 1.16 15.56
0.00 -0.25 1.00 0.80 0.00 0.20 17.20];

New solution is not optimal
Coefficient of x_5 is negative

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	**1.00**	0.00	0.00	12.20	0.00	-3.20	199.80
x_1	**0.00**	1.00	0.00	0.80	0.00	-0.80	7.20
x_4	**0.00**	0.00	0.00	-2.16	1.00	1.16	15.56
x_2	**0.00**	0.00	1.00	1.00	0.00	0.00	19.00

New solution is: x_1 = 7.2, **x_2 = 19.0**, x_3 = 0, x_4 =15.56 and x_5 = 0.
Value of the objective function **Z = 199.8**.

Marching to 4th Table

1. Select column x_5 as the Pivot column
2. Take ratio test and select third row as the Pivot row
3. Perform row operations to eliminate all coefficients of Pivot column (except the coefficient at the intersection of Pivot row and Pivot column)

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS	Ratio test
Z	**1.00**	0.00	0.00	12.20	0.00	-3.20	199.80	
x_1	**0.00**	1.00	0.00	0.80	0.00	-0.80	7.20	-9.00
x_4	**0.00**	0.00	0.00	-2.16	1.00	1.16	15.56	13.41
x_2	**0.00**	0.00	1.00	1.00	0.00	0.00	19.00	inf

Marching to 4th Table

1. To facilitate matters start doing row operations on the third row to make coefficient at the intersection of the Pivot row and Pivot column equal to one
2. Divide row (3) by 1.16
3. Now proceed with row operations for the remaining rows

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	**1.00**	0.00	0.00	12.20	0.00	-3.20	199.80
x_1	**0.00**	1.00	0.00	0.80	0.00	-0.80	7.20
x_4	**0.00**	0.00	0.00	-1.86	0.86	1.00	13.41
x_2	**0.00**	0.00	1.00	1.00	0.00	0.00	19.00

Fourth Table (Optimal Solution)

```
a = [1.00    0.00    0.00    6.24    2.76    0.00    242.72
     0.00    1.00    0.00   -0.69    0.69    0.00     17.93
     0.00    0.00    0.00   -1.86    0.86    1.00     13.41
     0.00    0.00    1.00    1.00    0.00    0.00     19.00];
```

New solution is optimal since all coefficients in Z-row are positive or zero

Basic Variable	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	**1.00**	0.00	0.00	6.24	2.76	0.00	242.72
x_1	**0.00**	1.00	0.00	-0.69	0.69	0.00	17.93
x_5	**0.00**	0.00	0.00	-1.86	0.86	1.00	13.41
x_2	**0.00**	0.00	1.00	1.00	0.00	0.00	19.00

New solution is: x_1 = 17.93, **x_2 = 19**, x_3 = 0, x_4 =13.41 and x_5 = 0.
Value of the objective function **Z = 242.72**.

Graphical Solution

Simplex method moves from corner point to corner point

Only corner points need to be investigated for optimality