Applications of Linear Programming - Minimization

Drs. Antonio A. Trani and H. Baik
Civil and Environmental Engineering
Virginia Tech

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Recall the Standard LP Form

Maximize $\sum_{j=1}^{n} c_j x_j$

subject to: $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ for $i = 1, 2, \ldots, m$

$x_j \ge 0$ for $j = 1, 2, \ldots, n$

Suppose that now we wish to investigate how to handle Minimization problems:

Minimization Problem Formulation

Minimize $$z = \sum_{j=1}^{n} c_j x_j$$

Reformulate as:

Maximize $$-z = -\sum_{j=1}^{n} c_j x_j$$

Minimization LP Example

A construction site requires a minimum of 10,000 cu. meters of sand and gravel mixture. The mixture must contain no less than 5,000 cu. meters of sand and no more than 6,000 cu. meters of gravel.

Materials may be obtained from two sites: 30% of sand and 70% gravel from site 1 at a delivery cost of $5.00 per cu. meter and 60% sand and 40% gravel from site 2 at a delivery cost of $7.00 per cu. meter.

a) Formulate the problem as a LP model

b) Solve using linprog and hand calculations

Minimization Problem

Min $z = 5x_1 + 7x_2$

s.t. $0.3x_1 + 0.6x_2 \geq 5000$ sand constraint

$0.7x_1 + 0.4x_2 \leq 6000$ gravel constraint

$x_1 + x_2 \geq 10000$ total amount of mixture

and $x_1 \geq 0$ $x_2 \geq 0$ non-negativity constraints

NOTE: x_1 and x_2 represent the amounts of material retrieved from sites 1 and 2 respectively.

Reformulation Steps

Max $-z = -5x_1 - 7x_2$ or

Max $-z + 5x_1 + 7x_2 + Mx_4 + Mx_7 = 0$

we want to ensure that artificial variables x_5 and x_7 are not part of the solution (use the BIG M or penalty method)

s.t. $0.3x_1 + 0.6x_2 - x_3 + x_4 = 5000$

Add a negative slack and a positive artificial variable

$0.7x_1 + 0.4x_2 + x_5 = 6000$

Add a slack variable

$x_1 + x_2 - x_6 + x_7 = 10000$

Add a negative slack and a positive artificial variable

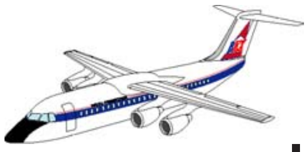
For each artificial variable added penalize the objective function with a Big-M

Conversion to Std. Format

Express the objective function with z -row coefficients for artificial variables to be zero. Thus we need to eliminate the z-row coefficents of x_4 and x_7.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
	[-1	5	7	0	M	0	0	M	0]
-M[	0	.3	.6	-1	1	0	0	0	5000]
-M[	0	1	1	0	0	0	-1	1	10000]
	[-1	(-1.3M+5)	(-1.6M+7)	M	0	0	M	0	-15000M]

Minimization Problem (Initial Tableau)



BV	z	x_1	x_2	x_3	$\bar{x}_4$	x_5	x_6	$\bar{x}_7$	RHS
z	**-1**	**-1.3M+5**	**-1.6M+7**	**M**	**0**	**0**	**M**	**0**	**-15000M**
x_4	0	0.3	0.6	-1	1	0	0	0	5000
$\mathbf{x_5}$	0	0.7	0.4	0	0	1	0	0	6000
x_7	0	1	1	0	0	0	-1	1	10000

IBFS = $(x_1, x_2, x_3, \bar{x}_4, x_5, x_6, x_7) = (0,0,0,5000,6000,0,10000)$

x_2 enters the basis (BV set) and x_4 (artificial) leaves

Minimization Problem (Second Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	$\bar{x}_7$	RHS
z	**-1**	**-0.5M+1.5**	**0**	**-1.67M -11.6**	**2.67M-11.67**	**0**	**M**	**0**	**-1666.4M-58333.4**
x_2	0	1/2	1	-5/3	5/3	0	0	0	8333.4
x_5	0	1/2	0	2/3	-2/3	1	0	0	2666.7
$\bar{x}_7$	0	1/2	0	5/3	-5/3	0	0	1	1666.7

Note: x_7 leaves and x_3 enters the basis

Minimization Problem (Third Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	-1								**70000**
x_2	0								10000
x_5	0								2000
x_3	0								1000

Note: x_3 leaves the basis and x_1 enters the basis

Minimization Problem

Note:Optimal solution (to be completed by students at home to make sure that we all understand the LP problem solutions using the Simplex Method)

Note: Four tables are needed to solve the problem

Mix Problem (Matlab Solution)

```
% Mixture Problem (gravel and sand materials)
% Enter the data:

  minmax=1;        % minimizes the objective function
  a=[0.3 0.6 -1 1 0 0 0
     0.7 0.4  0 0 1 0 0
     1.0 1.0  0 0 0 -1 1]
  b=[5000 6000 10000]'
  c=[-5 -7 0 -1000 0 0 -1000]  % I used -1000 for Big M
  bas=[4 5 7]
```

Matlab Solution (Using Linprog)

This phase is completed   - current basis is:

bas =
    2   5   1

The current basic variable (BV) values are :

b =
   1.0e+03 *
    6.6667   % for variable x2
    1.0000   % for variable x5
    3.3333   % for variable x1

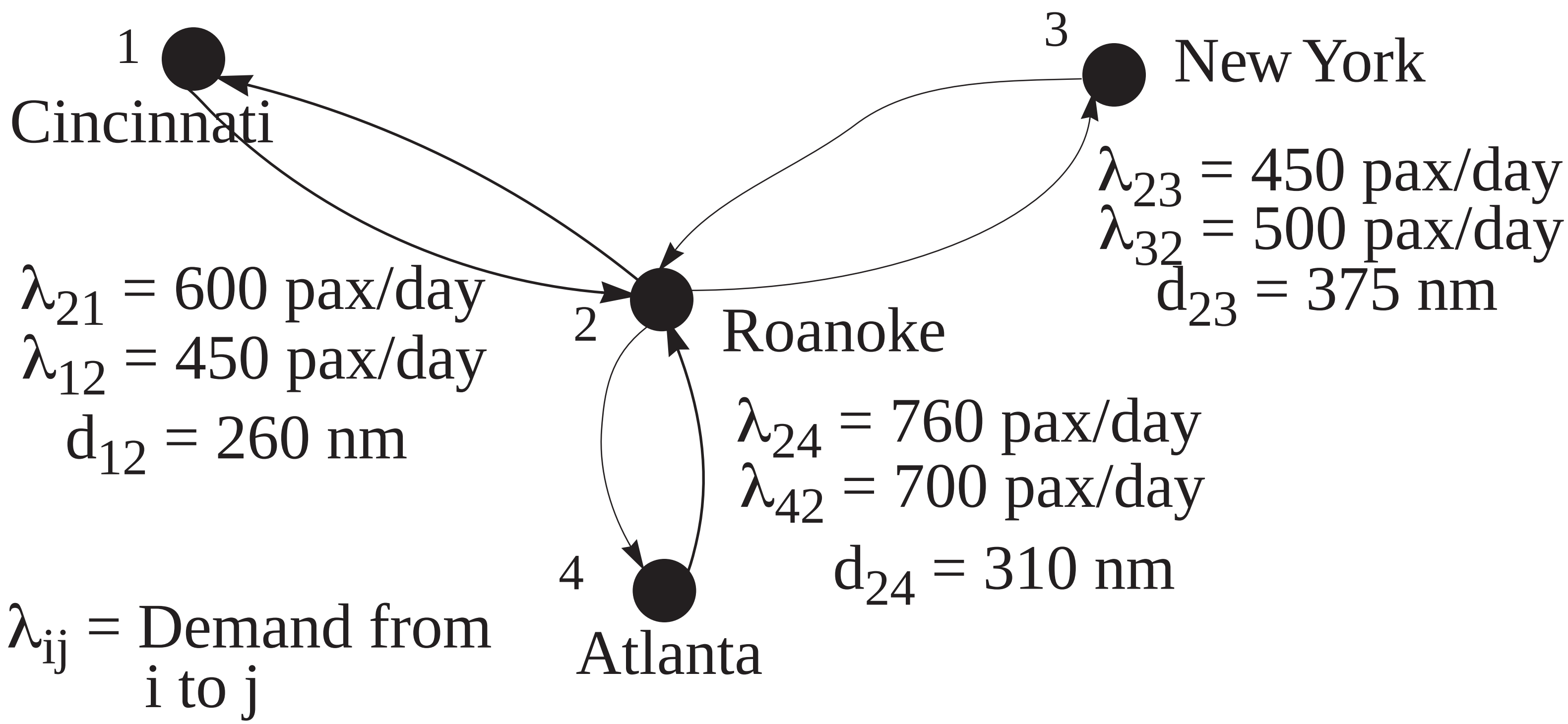
The current objective value is:
ans =
6.3333e+04 % dollars

The solution shows that optimally we need to buy 6,667 cu. meters of material from site 2 (x_2) and 3,333 cu. meters of material from site 1 (x_1). The total amount of material is 10,000 cu. meters as needed. The total cost is 63,334 dollars ($z = 7*6,666 + 5*3,333$).

Verify the solution by hand and plot the graphical solution as well (to do at home).

Airline Scheduling Problem (ASP-1)

A small airline would like to use mathematical programming to schedule its flights to maximize profit. The following map shows the city pairs to be operated.

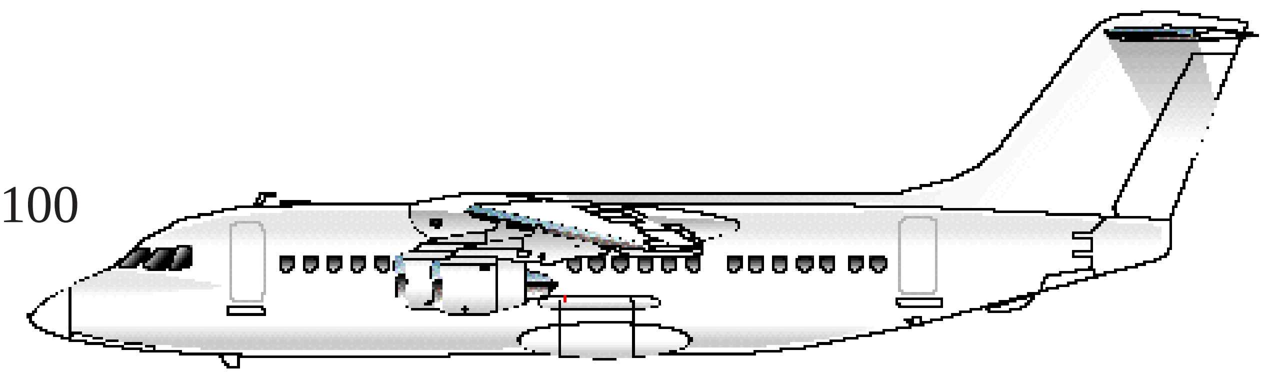


Airline Scheduling Problem

The airline has decided to purchase two types of aircraft to satisfy its needs: 1) the Embraer 145, a 45-seat regional jet, and 2) the Avro RJ-100, a four-engine 100 seater aircraft (see the following figure).

EMB-145

Avro RJ-100



Aircraft Characteristics

The following table has pertinent characteristics of these aircraft.

Aircraft	**EMB-145**	**Avro RJ-100**
Seating capacity - n_k	50	100
Block speed (knots) - v_k	400	425
Operating cost ($/hr) - c_k	1,850	3,800
Maximum aircraft utilization (hr/day)[a] - U_k	13.0	12.0

a. The aircraft utilization represents the maximum number of hours an aircraft is in actual use with the engines running (in airline parlance this is the sum of all daily block times). Turnaround times at the airport are not part of the utilization variable as defined here.

Nomenclature

Define the following sets of decision variables:

No. of acft. of type k in fleet = A_k

No. flights assigned from i to j using aircraft of type k = N_{ijk}

Minimum flight frequency between i and j = $(N_{ij})_{min}$

Based on expected load factors, the tentative fares between origin and destination pairs are indicated in the following table.

City pair designator	Origin-Destination	Average one-way fare ($/seat)
ROA-CVG	Roanoke to Cincinnati	175.00
ROA-LGA	Roanoke to La Guardia	230.00
ROA-ATL	Roanoke to Atlanta	200.00

Problem # 1 ASP-1 Formulation

1) Write a mathematical programming formulation to solve the ASP-1 Problem with the following constraints:

Maximize **Profit**

subject to:

- aircraft availability constraint
- demand fulfillment constraint
- minimum frequency constraint

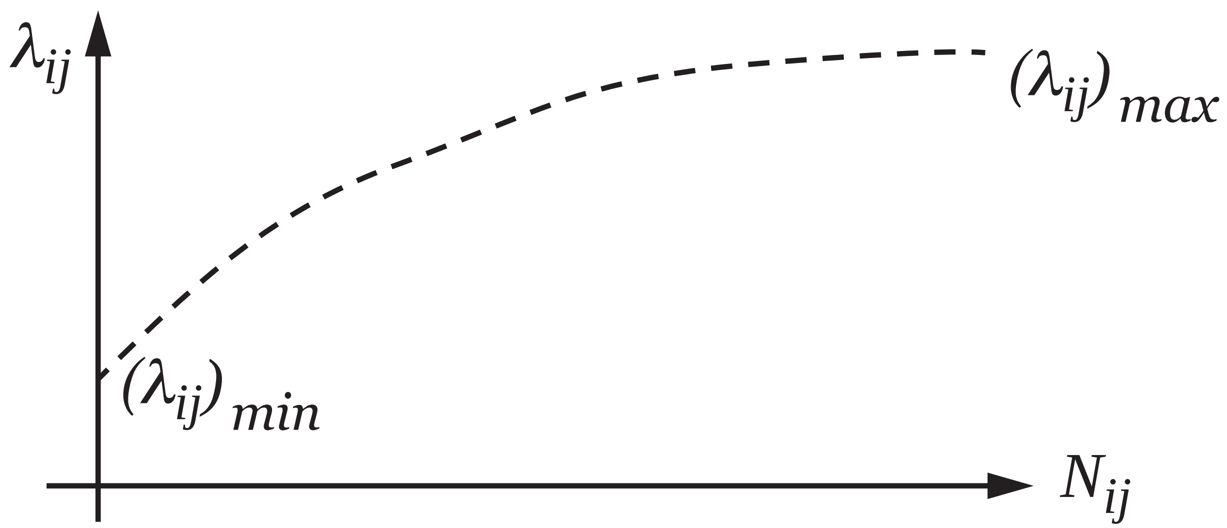
Problem # 2 ASP-1 Solution

1) Solve problem ASP-1 under the following numerical assumptions:

a) Maximize profit solving for the fleet size and frequency assignment without a minimum frequency constraint. Find the number of aircraft of each type and the number of flights between each origin-destination pair to satisfy the two basic constraints (demand and supply constraints).

b) Repeat part (a) if the minimum number of flights in the arc ROA-ATL is 8 per day (8 more from ATL-ROA) to establish a shuttle system between these city pairs.

c) Suppose the demand function λ_{ij} varies according to the number of flights scheduled between city pairs (see the following illustration). Reformulate the problem and explain (do not solve) the best way to reach an optimal solution.



Vehicle Scheduling Problem

Formulation of the problem.

Maximize **Profit**

subject to: (possible types of constraints)

a) aircraft availability constraint

b) demand fulfillment constraint

c) Minimum frequency constraint

d) Landing restriction constraint

Vehicle Scheduling Problem

Profit Function

P = Revenue - Cost

Revenue Function

$$\text{Revenue} = \sum_{(i,j)} \lambda_{ij} f_{ij}$$

where: λ_{ij} is the demand from i to j (daily demand)

f_{ij} is the average fare flying from i to j

Vehicle Scheduling Problem

Cost function

let N_{ijk} be the flight frequency from i to j using aircraft type k

let C_{ijk} be the total cost per flight from i to j using aircraft k

Cost = $\sum_{(i,j)} \sum_{k} N_{ijk} C_{ijk}$

then the profit function becomes,

Profit = $$\sum_{i,j} \lambda_{ij} f_{ij} - \sum_{i,j} \sum_{k} N_{ijk} C_{ijk}$$

Vehicle Scheduling Problem

Demand fulfillment constraint

Supply of seats offered > Demand for service

$$\sum_k n_k N_{ijk} \geq \lambda_{ij}$$ for all (i, j) city pairs or alternatively

$$\sum_k (lf) n_k N_{ijk} \geq \lambda_{ij}$$ for all (i, j) city pairs

lf is the load factor desired in the operation (0.8-0.85)

Note: airlines actually overbook flights so they usually factor a target load factor in their schedules to account for some slack

Vehicle Scheduling Problem

Aircraft availability constraint

(block time) (no. of flights) < (utilization)(no. of aircraft)

$$\sum_{(i,j)} t_{ijk} N_{ijk} \leq U_k A_k$$

one constraint equation for every k aircraft type

Vehicle Scheduling Problem

Minimum frequency constraint

No. of flights between i and j > Minimum number of desired flights

$\sum_k N_{ijk} \geq (N_{ij})_{min}$ for all (i, j) city pairs

Note: Airlines use this strategy to gain market share in highly traveled markets

Vehicle Scheduling Problem

Maximize Profit = $$\sum_{i,j} \lambda_{ij} f_{ij} - \sum_{i,j} \sum_{k} N_{ijk} C_{ijk}$$

subject to

$$\sum_{k} n_k N_{ijk} \geq \lambda_{ij}$$ for all (i, j) city pairs

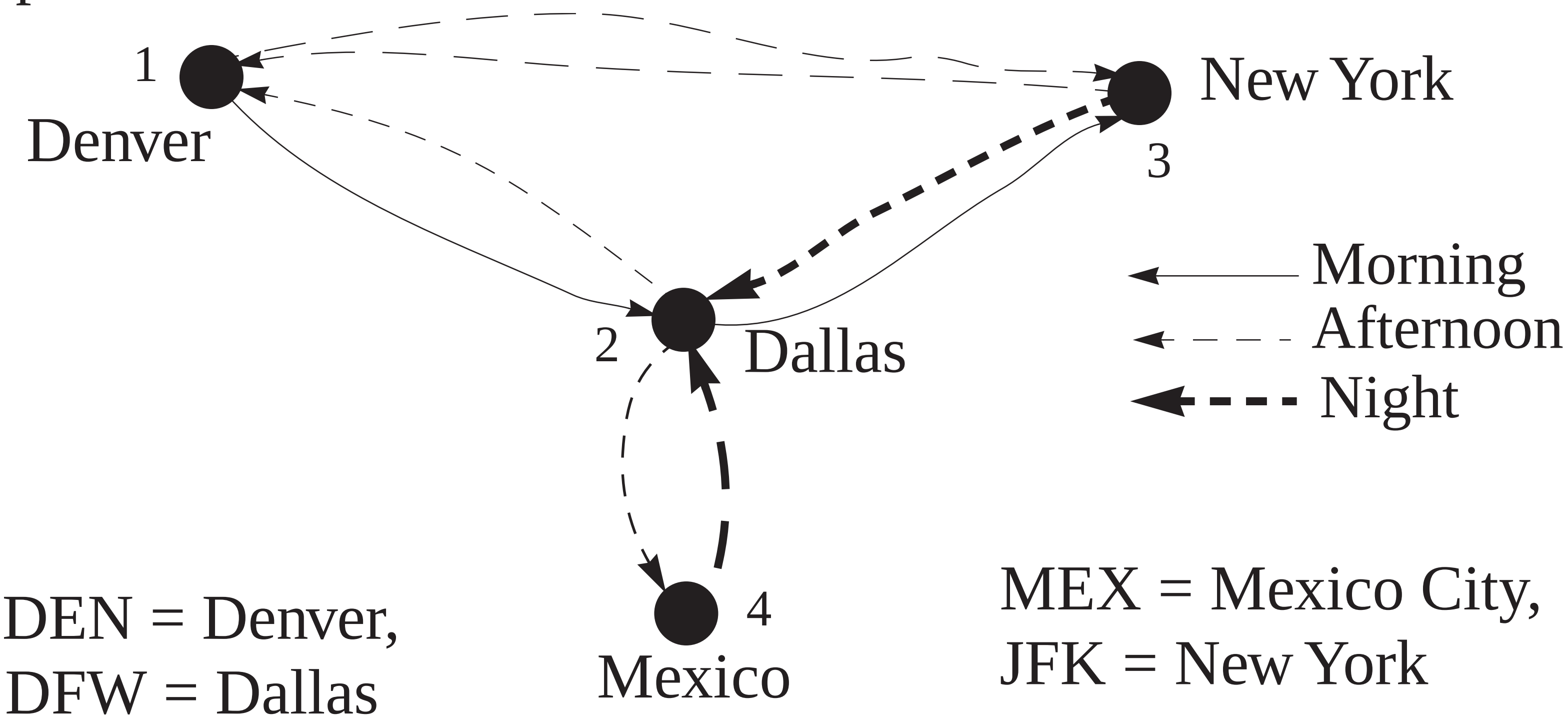
$$\sum_{(i,j)} t_{ijk} N_{ijk} \leq U_k A_k$$ for every k aircraft type

$$\sum_{k} N_{ijk} \geq (N_{ij})_{min}$$ for all (i, j) city pairs

Add Parts of the Solution Here

Crew Scheduling Problem

A small airline uses LP to allocate crew resources to minimize cost. The following map shows the city pairs to be operated.



DEN = Denver,
DFW = Dallas

MEX = Mexico City,
JFK = New York

Crew Scheduling Problem.

Flight Number	O-D Pair	Time of Day
100	DEN-DFW	Morning
200	DFW-DEN	Afternoon
300	DFW-MEX	Afternoon
400	MEX-DFW	Night
500	DFW-JFK	Morning
600	JFK-DFW	Night
700	DEN-JFK	Afternoon
800	JFK-DEN	Afternoon

Crew Scheduling Problem

Definition of terms:

a) Rotations consists of 1 to 2 flights (to make the problem simple)

b) Rotations cost $2,500 if terminates in the originating city

c) Rotations cost $3,500 if terminating elsewhere

Example of a feasible rotations are (100, 200), (500,800),(500), etc.

Crew Scheduling Problem

R_i	Single Flight Rotations	Cost ($)	R_i	Two-flight Rotations	Cost ($)
1	100	3,500	9	100,200	2,500
2	200	3,500	10	100,300	3,500
3	300	3,500	11	500,800	3,500
4	400	3,500	12	500,600	2,500
5	500	3,500	13	300,400	2,500
6	600	3,500	14	200,100	3,500
7	700	3,500	15	600,300	3,500

R_i	Single Flight Rotations	Cost ($)	R_i	Two-flight Rotations	Cost ($)
8	800	3,500	16	600,200	3,500
			17	600,500	3,500
			18	800,100	3,500
			19	700,600	3,500
			20	700,800	3,500

Decision variables:

$$R_i = \begin{cases} 1 & \text{if } i \text{ rotation is used} \\ 0 & \text{if } i \text{ rotation is not used} \end{cases}$$

Crew Scheduling Problem

Min **Cost**

subject to: (possible types of constraints)

a) each flight belongs to a rotation (to a crew)

Min

z = 3500 R_1 + 3500 R_2 + 3500 R_3 + 3500 R_4 + 3500 R_5 +

3500 R_6 + 3500 R_7 + 3500 R_8 + 2500 R_9 + 3500 R_{10}

$3500\ R_{11} + 2500\ R_{12} + 2500\ R_{13} + 3500\ R_{14} + 3500\ R_{15}$

$3500\ R_{16} + 3500\ R_{17} + 3500\ R_{18} + 3500\ R_{19} + 3500\ R_{20}$

s.t. (Flt. 100) $R_1 + R_9 + R_{10} + R_{14} + R_{18} = 1$

(Flt. 200) $R_2 + R_9 + R_{14} + R_{16} = 1$

(Flt. 300) $R_3 + R_{10} + R_{13} + R_{15} = 1$

(Flt. 400) $R_4 + R_{13} = 1$

(Flt. 500) $R_5 + R_{11} + R_{12} + R_{17} = 1$

(Flt. 600) $R_6 + R_{12} + R_{15} + R_{16} + R_{17} + R_{19} = 1$

(Flt. 700) $R_7 + R_{19} + R_{20} = 1$

(Flt. 800) $R_8 + R_{11} + R_{18} + R_{20} = 1$

Crew Scheduling Problem

Problem statistics:

a) 20 decision variables (rotations)

b) 8 functional constraints (one for each flight)

c) All constraints have equality signs

Crew Scheduling Problem (Matlab)

Input File

```
minmax=1; % Minimization formulation
a=[1 0 0 0 0 0 0 0 1 1 0 0 0 1 0 0 0 1 0 0
   0 1 0 0 0 0 0 0 1 0 0 0 0 1 0 1 0 0 0 0
   0 0 1 0 0 0 0 0 1 0 0 1 0 1 0 0 0 0 0 0
   0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
   0 0 0 0 1 0 0 0 0 0 1 1 0 0 0 0 0 0 1 0
   0 0 0 0 0 1 0 0 0 0 0 1 0 0 1 1 1 0 1 0
   0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 1
   0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 1 0 1]
b=[1 1 1 1 1 1 1 1]'
```

c=[-3500 -3500 -3500 -3500 -3500 -3500 -3500
-3500 -2500 -3500 -3500 -2500 -2500 -3500
-3500 -3500 -3500 -3500 -3500 -3500]

bas=[1 2 3 4 5 6 7 8]

Optimal Solution (after 8 iterations):

bas =[9 12 13 4 19 18 20 11]

The current basic variable values are :

b =1	rotation 9 (100,200)	Cost = $2,500
1	rotation 12 (500,600)	Cost = $2,500
1	rotation 13 (300,400)	Cost = $2,500
0	rotation 4 (400)	
0	rotation 19 (700,600)	
0	rotation 18 (800,100)	
1	rotation 20 (700,800)	Cost = $3,500
0	rotation 11 (500,800)	

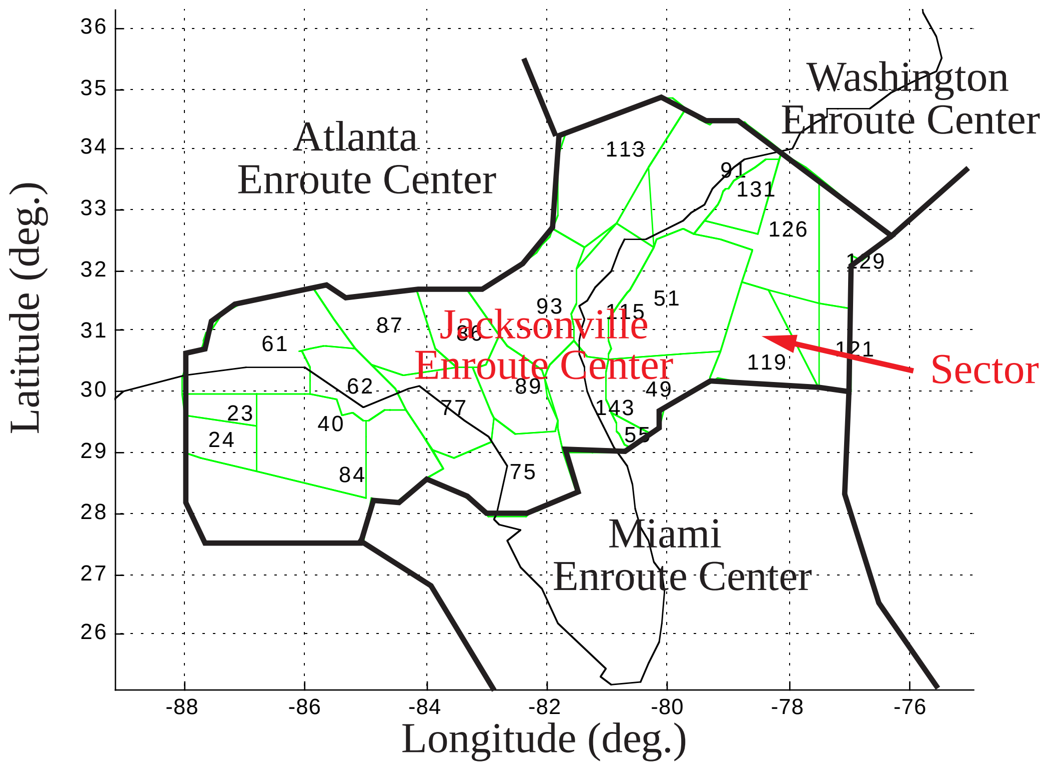
z = $11,000 dollars to complete all flights (4 crews assigned)

Human Resource Assignment Problem (ATC Application)

Linear programming problems are quite useful for solving staffing problems where human resources are typically scheduled over periods of varying activities. Consider the case of the staffing requirements of a busy Air Route Traffic Control Center (ARTCC) where Air Traffic Control (ATC) personnel monitor and direct flights over large regions of airspace in the Continental U.S. Given that traffic demands vary over the time of day ATC controller staffing requirements vary as well. Take for example Jacksonville ARTCC comprised of 35 sector boundaries (see Figure 1).

Each sector is managed by one or more controllers depending on the traffic load.

Jacksonville ARTCC Sectorization at 40,000 ft.



Relevant Questions

A task analysis study estimates the staffing requirements for this ARTCC (see Table 1). Let x_i be the number of ATC controllers that start their workday during the ith hour (, x_2, …,).

a) Formulate this problem as a linear programming problem to find the least number of controllers to satisfy the staffing constraints based on traffic demands expected at this FAA facility. Assume controllers work shifts of 8 hours (no overtime is allowed for now).

b) Write the objective function of the **first Simplex tableau** to solve this problem.

c) Find the **minimum number of controllers** needed to satisfy the staffing requirements using linprog. Comment on the solution.

d) Human factors studies suggest ATC controllers take one hour of rest during their 8-hour work period to avoid excessive stress. The ATC manager at this facility instructs all personnel to take the one-hour rest period after working four consecutive hours. Reformulate the problem and find the new optimal solution.

e) The average salary for ATC personnel is $65,000 for normal operation hours (5:00 -19:00 hours) with a 15% higher compensation for those working the night shift (19:00 until 5:00 hours). **Reformulate** the problem to allocate ATC controllers to minimize the cost of the operation. Assume the one-hour break rule applies.

TABLE 1. Expected Staffing Requirements at Jacksonville ARTCC Center (Jacksonville, FL).

Time of Day (EST)	Staff Needs	Remarks
0:00 - 2:00	30	Light traffic
2:00 - 5:00	25	Light traffic - few airline flights
5:00 - 7:00	35	Moderate traffic
7:00 - 10:00	48	Heavy traffic (morning “push”)
10:00 - 13:00	35	Moderate traffic
13:00 - 17:00	31	Moderate traffic
17:00 - 21:00	42	Heavy evening traffic
21:00 - 24:00	34	Moderate traffic