CEE 3804 - Computer Applications

Mathematical Programming (LP)

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Resource Allocation

Principles of **Mathematical Programming**

Mathematical programming is a general technique to solve resource allocation problems using optimization. Types of problems:

- Linear programming
- Integer programming
- Dynamic programming
- Decision analysis
- Network analysis and CPM

Mathematical Programming

Operations research was born with the increasing need to solve optimal resource allocation during WWII.

- Air Battle of Britain
- North Atlantic supply routing problems
- Optimal allocation of military convoys in Europe

Dantzig (1947) is credited with the first solutions to linear programming problems using the Simplex Method

Resource Allocation

Linear Programming Applications

- Allocation of products in the market
- Mixing problems
- Allocation of mobile resources in infrastructure construction (e.g., trucks, loaders, etc.)
- Crew scheduling problems
- Network flow models
- Pollution control and removal
- Estimation techniques

Linear Programming

General Formulation

Maximize $$\sum_{j=1}^{n} c_j x_j$$

subject to: $$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for} \quad i = 1, 2, \ldots, m$$

$$x_j \ge 0 \ \text{for} \quad j = 1, 2, \ldots, n$$

Linear Programming

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to:

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$

...

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$

and $x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$

Linear Programming

$$\sum_{j=1}^{n} c_j x_j$$ Objective Function (OF)

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$ Functional Constraints (*m* of them)

$x_j \geq 0$ Nonnegativity Conditions (*n* of these)

x_j are decision variables to be optimized (min or max)

c_j are costs associated with each decision variable

Linear Programming

a_{ij} are the coefficients of the functional constraints

b_i are the amounts of the resources available (RHS)

Some definitions

Feasible Solution (FS) - A solution that satisfies all functional constraints of the problem

Basic Feasible Solution (BFS)- A solution that needs to be further investigated to determine if optimal

Initial Basic Feasible Solution - a BFS used as starting point to solve the problem

LP Example (Construction)

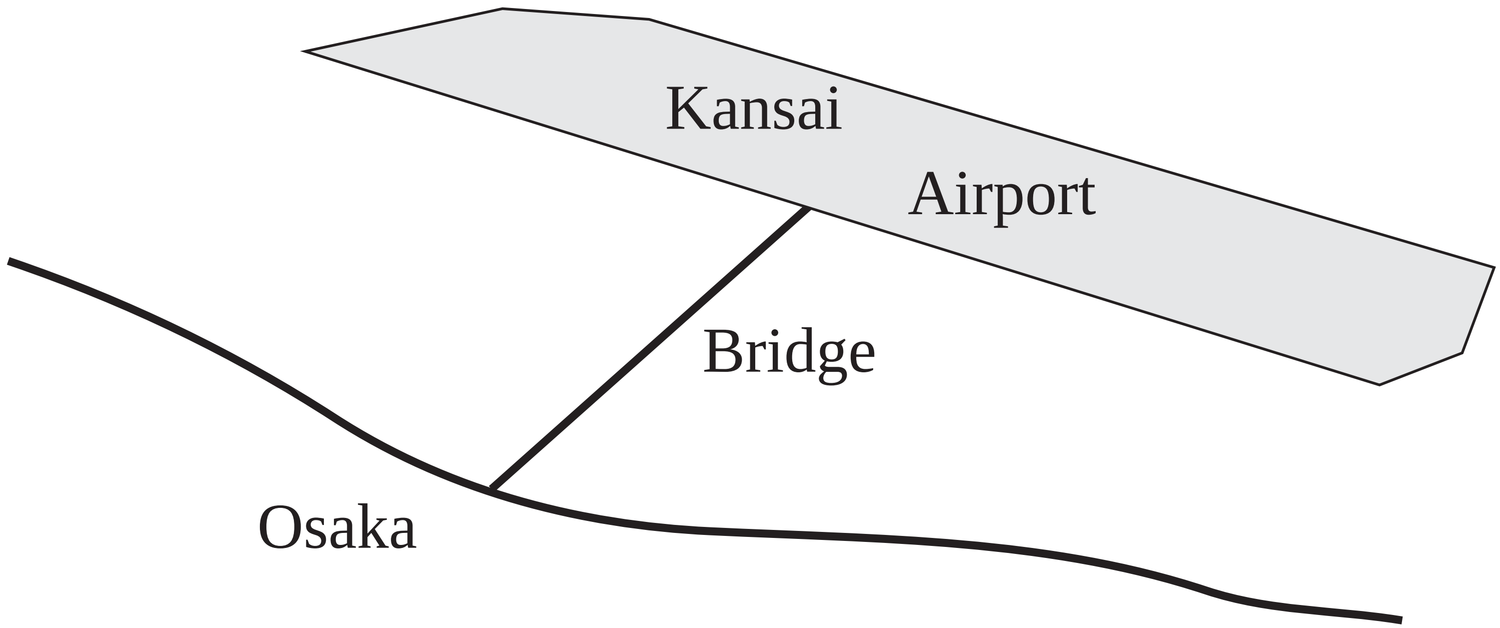
During the construction of an off-shore airport in Japan the main contractor used two types of cargo barges to transport materials from a fill collection site to the artificial island built to accommodate the airport.

The types of cargo vessels have different cargo capacities and crew member requirements as shown in the table:

Vessel Type	**Capacity (m-ton)**	**Crew required**	**Number available**
Fuji	300	3	40
Haneda	500	2	60

Osaka Bay Model

According to company records there are 180 crew members in the payroll and all crew members are trained to either manage the "Haneda" or "Fuji" vessels.



Osaka Bay Model

Mathematical Formulation

Maximize $Z = 300x_1 + 500x_2$

subject to: $3x_1 + 2x_2 \leq 180$

$x_1 \leq 40$

$x_2 \leq 60$

$x_1 \geq 0$ and $x_2 \geq 0$

Note: let x_1 and x_2 be the no. “Fuji” and “Haneda” vessels

Osaka Bay LP Model

Maximize $Z = 300x_1 + 500x_2$

Solution:

a) Covert the problem to standard (canonical) form

subject to:

$$3x_1 + 2x_2 + x_3 = 180$$

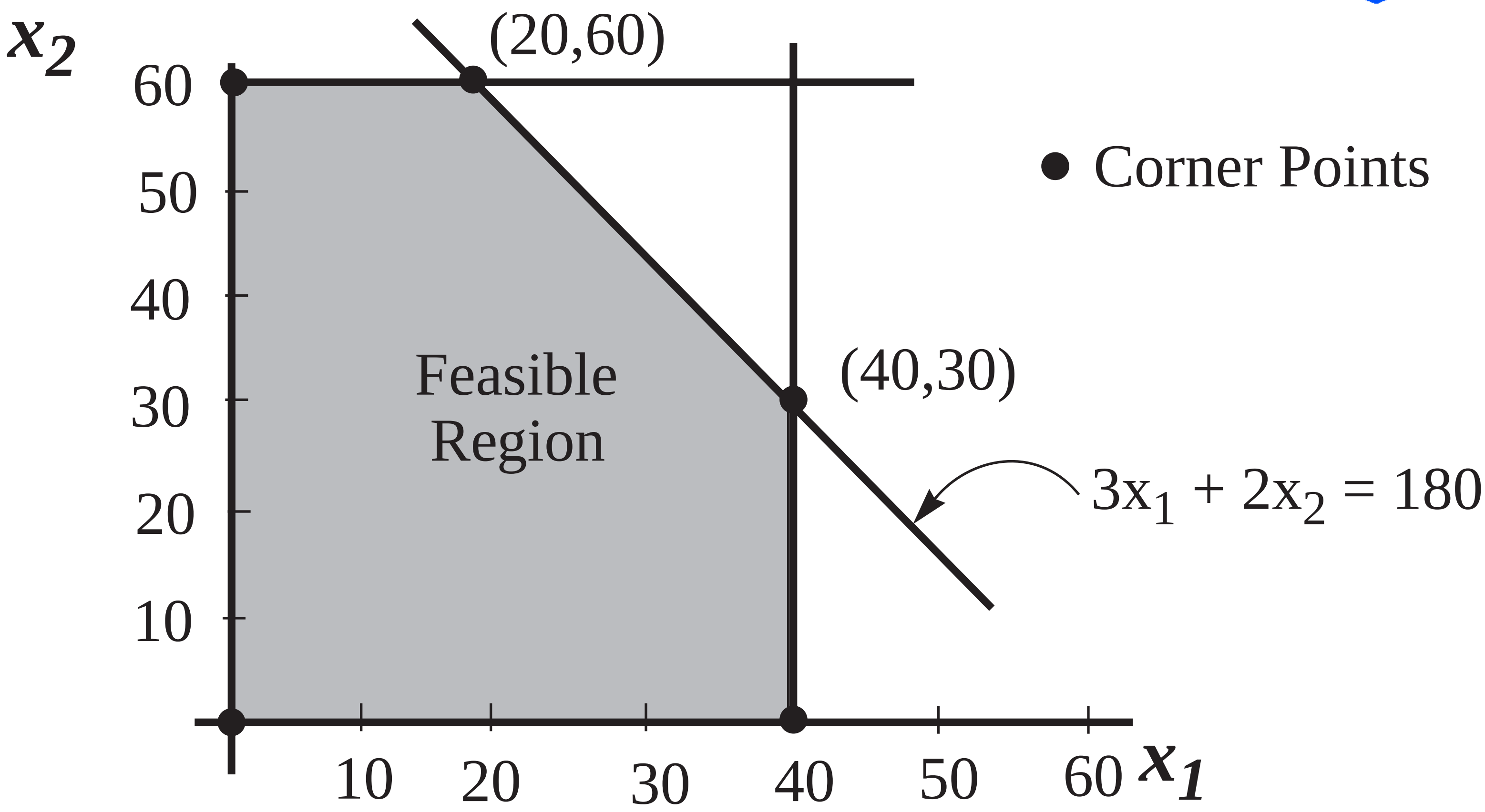
$$x_1 + x_4 = 40$$

$$x_2 + x_5 = 60$$

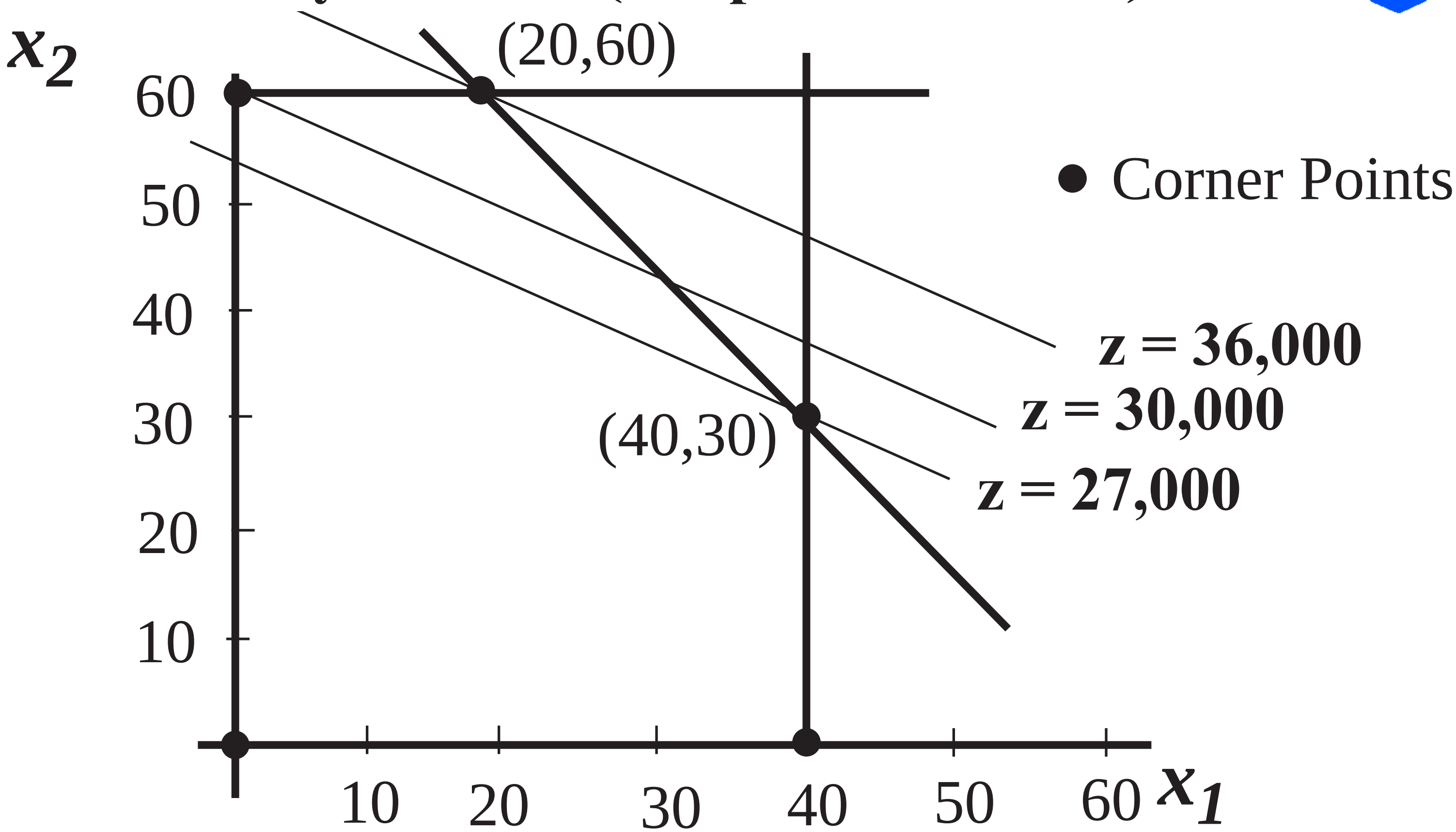
$x_1 \geq 0$ and $x_2 \geq 0$

Add a slack variable for each **<=** type constraint

Osaka Bay Problem (Graphical Solution)



Osaka Bay Problem (Graphical Solution)



Note: Optimal Solution $(x_1, x_2) = (20,60)$ vessels

Osaka Bay Problem (Simplex Method)

Arrange objective function in standard form to perform Simplex tableaus

$$Z - 300x_1 - 500x_2 = 0$$

$$3x_1 + 2x_2 + x_3 = 180$$

$$x_1 + x_4 = 40$$

$$x_2 + x_5 = 60$$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Note: x_3, x_4, x_5 are **slack variables**

Osaka Bay Example (Initial Tableau)

BV	**z**	x_1	x_2	x_3	x_4	x_5	**RHS**
z	**1**	**-300**	**-500**	**0**	**0**	**0**	**0**
x_3	0	3	2	1	0	0	180
x_4	0	1	0	0	1	0	40
x_5	0	0	1	0	0	1	60

BV = x_3, x_4, x_5 and NBV = x_1, x_2

BV = Basic Variable (non-zero)

NBV = Non-basic variable (zero)

Simplex Method Procedure

- Examine the objective function in the current Tableau
  - If the coefficients of the non-basic variables (i.e., those which are zero in the current solution) are negative, the value of the objective function can still be improved by introducing one of the NBVs to the solution set
  - Select the most negative coefficient value of the NBV in the Z-row and introduce that NVB to the solution
  - Allocate as much of the variable selected until the constraint equations limit the value of the NVB introduced

Simplex Method

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-300	-500	0	0	0	0
x_3	0	3	2	1	0	0	180
x_4	0	1	0	0	1	0	40
x_5	0	0	1	0	0	1	60

BV = x_3, x_4, x_5 and NBV = x_1, x_2

Most negative coefficient in Z-row improves the value of Z the most x_2 is selected as the NVB that will be introduced to the BV set in the next iteration

Select the column of variable x_2 as "pivot" column for calculations in the next Tableau

Simplex Method

- Now we know x_2 will be part of the solution in the next Tableau
- The question is which one of the BV variables (x_3, x_4 and x_5) will leave the solution (the so-called Basis)
  - Examine the constraint equations to make that decision
  - The variable that leaves the BV set is that one that first becomes zero when x_2 is increased

Simplex Method: Check Constraints

- From the original constraint equations

Recall in the current solution x_1 is zero

$$3x_1 + 2x_2 + x_3 = 180$$

x_2 can be as high as 90 before the constraint is violated

$$x_1 + x_4 = 40$$

x_2 is not part of this constraint, x_2 can be made as large as you want

$$x_2 + x_5 = 60$$

x_2 can be as high as 60 before the constraint is violated

Bottom Line: 3rd constraint equation limits the value of x_2 the most. x_5 leaves the solution and x_2 becomes a BV variable (non-zero)

Simplex Method: Check Constraints

- The selection of the leaving BV variable can be simplified using the ratio test

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
z	1	-300	-500	0	0	0	0	
x_3	0	3	2	1	0	0	180	90
x_4	0	1	0	0	1	0	40	inf
x_5	0	0	1	0	0	1	60	60

Ratio of RHS of constraint equation and the coefficient of the variable in pivot column (x_2 in this table)

For row x_3 : 180/2 = 90
For row x_4 : 40/0 = infinity
For row x_5 : 60/1 = 60

Bottom Line: 3rd constraint equation limits the value of x_2 the most. Select the pivot row as the row with the smallest ratio

Solution: (x_1, x_2,x_3, x_4, x_5) = (0,0,180,40,60)

Osaka Bay Example (Initial Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
z	**1**	**-300**	**-500**	**0**	**0**	**0**	**0**	
x_3	0	3	2	1	0	0	180	90
x_4	0	1	0	0	1	0	40	inf
x_5	0	0	1	0	0	1	60	60

x_2 improves the objective function more than x_1

Simplex Method: Pivot Row and Column

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**-500**	**0**	**0**	**0**	**0**
x_3	0	3	2	1	0	0	180
x_4	0	1	0	0	1	0	40
x_5	0	0	1	0	0	1	60

BV = x_3, x_4, x_5 and NBV = x_1, x_2

Select the row of variable x_5 as "pivot" row for calculations in the next Tableau

Select the column of variable x_2 as "pivot" column for calculations in the next Tableau

In the next Tableau, x_5 leaves the solution and x_2 is now a non-zero variable (part of the solution)

Simplex Method : Matrix Operations

- Developing the next Tableau requires a few linear algebra manipulations:
  - Zero out the coefficient of the Z-row for the pivot column chosen
  - Zero out all coefficients in the pivot row except for the coefficient at the intersection of the pivot row and pivot column
  - Do repeated linear algebra row operations to zero out every coefficient in the pivot column

Simplex Method : Matrix Operations

- Example: to zero out the coefficient (-500) in the Z-row of the pivot column
  - Multiply the entire row representing the 3rd constraint equation (x_5 row) in the Tableau by 500 and add to the Z-row

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**-500**	**0**	**0**	**0**	**0**
x_3	0	3	2	1	0	0	180
x_4	0	1	0	0	1	0	40
x_5	0	0	1	0	0	1	60

x 500 + Z-row

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**0**	**0**	**0**	**500**	**30,000**

Simplex Method : Matrix Operations

- To zero out the coefficient (2) in the x_3 row of the pivot column
  - Multiply the entire row representing the 3rd constraint equation (x_5 row) in the Tableau by -2 and add to x_3 row

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**-500**	**0**	**0**	**0**	**0**
x_3	0	3	2	1	0	0	180
x_4	0	1	0	0	1	0	40
x_5	0	0	1	0	0	1	60

x (-2)+ x_3 row

x_3	0	3	0	1	0	-2	60

Simplex Method : Matrix Operations

- The coefficient of row x4 is already zero so no further matrix algebra computations are needed
- The new Tableau is now ready to be assembled

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**0**	**0**	**0**	**500**	**30,000**
x_3	0	3	0	1	0	-2	60
x_4	0	1	0	0	1	0	40
x_2	0	0	1	0	0	1	60

Leaving BV = x_5 : New BV = x_2

Osaka Bay Example (Second Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
z	**1**	**-300**	**0**	**0**	**0**	**500**	**30,000**	
x_3	0	3	0	1	0	-2	60	20
x_4	0	1	0	0	1	0	40	40
x_2	0	0	1	0	0	1	60	inf

x_1 improves the objective function the maximum

Simplex Method: Check for Optimality Conditions

- Examine the objective function (Z-row) in the current Tableau
  - If the coefficients of the non-basic variables (i.e., those which are zero in the current solution) are negative, the value of the objective function can still be improved by introducing one of the NBVs to the solution set
  - Since the coefficient of x1 is negative, we conclude that the solution can be improved if we introduce x1 to the BV set
  - Repeat the steps in the previous slides

Simplex Method: Iterations

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**0**	**0**	**0**	**500**	**30,000**
x_3	0	3	0	1	0	-2	60
x_4	0	1	0	0	1	0	40
x_2	0	0	1	0	0	1	60

Most negative coefficient in Z-row improves the value of Z the most x_1 is selected as the NVB that will be introduced to the BV set in the next iteration

Select the column of variable x_1 as "pivot" column for calculations in the next Tableau

Simplex Method : Matrix Operations

- Select pivot row by taking the ratio test (smallest ratio)
- Row x3 is selected as the pivot row
- Multiply the entire x3 row by 1/3 to make the coefficient of the intersection cell unity

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
z	**1**	**-300**	**0**	**0**	**0**	**500**	**30,000**	
x_3	0	3	0	1	0	-2	60	20
x_4	0	1	0	0	1	0	40	40
x_2	0	0	1	0	0	1	60	inf

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
x_3	0	1	0	1/3	0	-2/3	20

Simplex Method : Matrix Operations

- Multiply pivot row by 300 and add to Z-row to zero out the (-300) coefficient in the pivot column
- Repeat the elimination for other rows in the pivot column

BV	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	-300	0	0	0	500	30000
x_3	0	1	0	1/3	0	-2/3	20
x_4	0	1	0	0	1	0	40
x_5	0	0	1	0	0	1	60

x_3 row x (300)+ Z row

BV	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	0	0	100	0	300	36000

Leaving BV = x_3 : New BV = x_1

Osaka Bay Example (Final Tableau)

BV	**z**	***x_1***	***x_2***	***x_3***	***x_4***	***x_5***	**RHS**
z	**1**	**0**	**0**	**100**	**0**	**300**	**36,000**
x_1	0	1	0	1/3	0	-2/3	20
x_4	0	0	0	-1/3	1	2/3	20
x_2	0	0	1	0	0	1	60

Note: All NVB coefficients are positive or zero in tableau

Optimal Solution: $(\boldsymbol{x_1}, \boldsymbol{x_2}, x_3, x_4, x_5) = (\mathbf{20}, \mathbf{60}, 0, 20, 0)$

Osaka Bay Model (Revised)

Mathematical Formulation

Maximize $Z = 300x_1 + 500x_2$

subject to: $3x_1 + 2x_2 = 180$ Revised Constraint

$x_1 \leq 40$

$x_2 \leq 60$

$x_1 \geq 0$ and $x_2 \geq 0$

Note: let x_1 and x_2 be the no. “Fuji” and “Haneda” vessels

Osaka Bay Model (Revised)

Maximize $Z = 300x_1 + 500x_2$

a) Covert the problem in standard form

subject to: $3x_1 + 2x_2 = 180$

$x_1 + x_3 = 40$

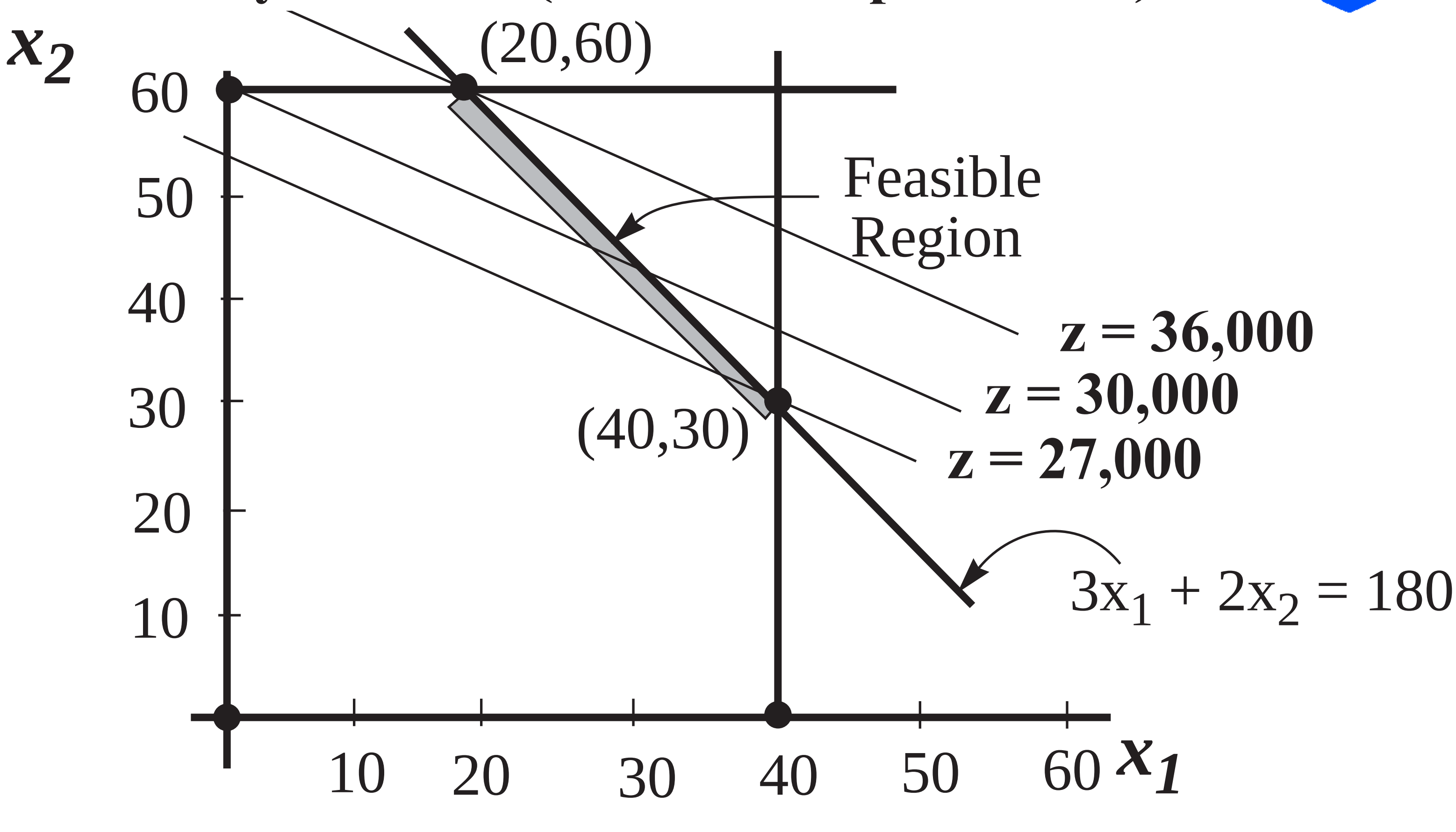
$x_2 + x_4 = 60$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and $x_4 \geq 0$

- Note: Problem lacks an intuitive IBFS (see first constraint)

- Note that setting $x_1 = 0$ and $x_2 = 0$ produces finite integer values for x_3 and x_4 (40 and 60, respectively) but fails to provide and adequate solution for constraint (1).
- This requires a reformulation step where another variable is added to the problem to identify an IBFS
- Add an artificial variable to the first constraint to solve the problem
- Adding an artificial variable in the constraint equation requires the addition of a large penalty to the objective function (z) to avoid this artificial variable being part of the solution

Osaka Bay Problem (Revised Graphical Sol.)



Osaka Bay Model (Revised)

Maximize $Z = 300x_1 + 500x_2$

a) Add an artificial variable to the initial “equal to” constraint

subject to: $3x_1 + 2x_2 + \bar{x}_5 = 180$

$$x_1 + x_3 = 40$$

$$x_2 + x_4 = 60$$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

IBFS is now evident with x_1 and x_2 being zero (NVB).

Revised Solution (Big-M Method)

Revise the **objective function** to drive artificial variable to zero in the optimal solution. M is a large positive number.

Maximize $Z = 300x_1 + 500x_2 - Mx_5$

subject to:

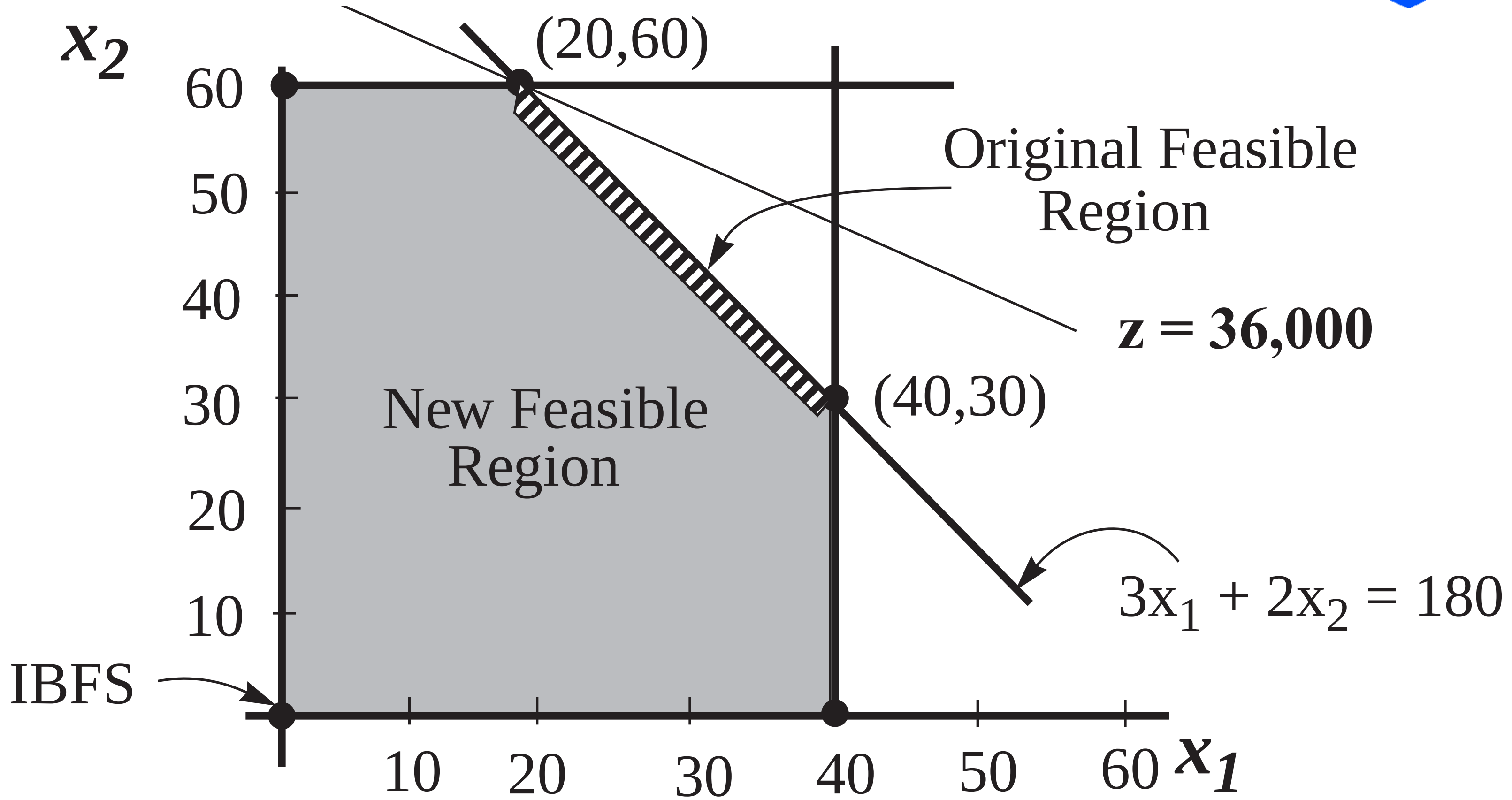
$$3x_1 + 2x_2 + \bar{x}_5 = 180$$

$$x_1 + x_3 = 40$$

$$x_2 + x_4 = 60$$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Osaka Bay LP (Expanded Feasible Region)



Revised Solution (Big-M Method)

Rearrange the OF and constraints before solving

Maximize $Z - 300x_1 - 500x_2 + Mx_5 = 0$

subject to: $x_1 + x_3 = 40$

$x_2 + x_4 = 60$

$3x_1 + 2x_2 + \bar{x}_5 = 180$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Note: the “Big M” (or a large penalty) is added to each artificial variable in OF. x_3 and x_4 are slack variables, x_5 is an artificial variable.

Revised Osaka Bay LP (Initial Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**-500**	**0**	**0**	**M**	**0**
x_3	0	1	0	1	0	0	40
x_4	0	0	1	0	1	0	60
x_5	0	3	2	0	0	1	180

BV = x_3, x_4, x_5 and NBV = x_1, x_2

Solution: (x_1, x_2,x_3, x_4, x_5) = (0,0,40,60,180)

Revised Osaka Bay LP (Initial Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	
z	**1**	**-3M-300**	**-2M-500**	**0**	**0**	**0**	**-180M**	
x_3	0	1	0	1	0	0	40	40
x_4	0	0	1	0	1	0	60	inf
x_5	0	3	2	0	0	1	180	60

x_1 improves the objective function the maximum

Leaving BV = x_3 : New BV = x_1

Revised Osaka Bay LP (2nd Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	
z	**1**	**0**	**-2M-500**	**3M+300**	**0**	**0**	**-60M+ 12000**	
x_1	0	1	0	1	0	0	40	inf
x_4	0	0	1	0	1	0	60	60
x_5	0	0	2	-3	0	1	60	30

x_2 improves the objective function the maximum. Leaving BV $= x_5$: New BV = x_2

Revised Osaka Bay LP (3rd Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	
z	**1**	**0**	**0**	**-450**	**M+250**	**0**	**27000**	
x_1	0	1	0	1	0	0	40	40
x_4	0	0	0	3/2	1	-1/2	30	20
x_2	0	0	1	-3/2	0	1/2	30	no

x_3 improves the objective function the maximum. Leaving BV = x_4 : New BV = x_3

Revised Osaka Bay LP (Final Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**0**	**0**	**0**	**300**	**M+100**	**36000**
x_1	0	1	0	0	-2/3	1/3	20
x_3	0	0	0	1	2/3	-1/3	20
x_2	0	0	1	0	-1/2	1/2	60

Note: All NVB coefficients are positive or zero in tableau

Optimal Solution: ($\boldsymbol{x_1}$, $\boldsymbol{x_2}$, x_3, x_4, x_5) = (**20**,**60**,20,0,0)

Simplex Method Anomalies

a) Ties for leaving BV - break without arbitration

b) Ties for entering BV - break without arbitration

c) Zero coefficient of NBV in OF (final tableau) - Implies multiple optimal solutions

d) No leaving BV - implies unbounded solution

Steps in the Simplex Method

I) Initialization Step

- Introduce slack variables
- Select original variables of the problems as part of the NBV
- Select slacks as BV

II) Stopping Rule

- The solution is optimal if every coefficient in the OF is nonnegative

- Coefficients of OF measure the rates of change of the OF as any other variable increases from zero

III) Iterative Step

- Determine the entering NBV (pivot column)
- Determine the leaving BV (from BV set) as the first variable to go to zero without violating constraints
- Perform row operations to make coefficients of BV unity in their respective rows
- Eliminate new BV coefficients (from pivot column) from other equations performing row operations

Linear Programming Strategies Using the Simplex Method

- Identify the problem
- Formulate the problem using LP
- Solve the problem using LP
- Test the model (correlation and sensitivity analysis)
- Establish controls over the model
- Implementation
- Model re-evaluation

LP Formulations

Type of Constraint	**How to handle**
$3x_1 + 2x_2 \le 180$	Add a slack variable
$3x_1 + 2x_2 = 180$	Add an artificial variable
	Add a penalty to OF (BigM)
$3x_1 + 2x_2 \ge 180$	Add a negative slack and a positive artificial variable

LP (Handling Constraints)

Type of Constraint	**Equivalent Form**
$3x_1 + 2x_2 \le 180$	$3x_1 + 2x_2 + x_3 = 180$
$3x_1 + 2x_2 = 180$	$3x_1 + 2x_2 + x_3 = 180$ $z = c_1x_1 + c_2x_2 - Mx_3$
$3x_1 + 2x_2 \ge 180$	$3x_1 + 2x_2 - x_3 + x_4 = 180$ $z = c_1x_1 + c_2x_2 - Mx_4$

Note: M is a large positive number

Theory Behind Linear Programming (per Hillier and Lieberman)

General Formulation

Maximize $Z = \sum_{j=1}^{n} c_j x_j$

subject to: $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ for $i=1,2, \ldots, m$

$x_j \ge 0$ for $j=1,2,\ldots,n$

General LP Formulation (Matrix Form)

Maximize $Z = \boldsymbol{cx}$

subject to: $\boldsymbol{Ax} = \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$ where:

$\boldsymbol{c}$ is the vector containing the coefficients of the O.F.,

$\boldsymbol{A}$ is the matrix containing all coefficients of the functional constraints,

$\boldsymbol{b}$ is the column vector for RHS coefficients,

$\boldsymbol{x}$ is the vector of decision variables

note that: $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 \ldots & c_n \end{bmatrix}$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}, \ \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ and matrix $\boldsymbol{A}$

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{bmatrix}$$

Theory Behind the Simplex Method

Addition of slack variables to the problem yields:

$$\boldsymbol{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ x_{n+m} \end{bmatrix}$$ where $\boldsymbol{x}_s$ is a vector of slack variables (m)

New augmented constraints become,

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \boldsymbol{b} \text{ and } \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} \geq \mathbf{0}$$

Note: I is an $m \times m$ identity matrix.

Theory Behind the Simplex Method

Basic Feasible Solution. From the system,

$[A \;\; I]\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \boldsymbol{b}$ n Nonbasic Variables (NBV) from the set,

$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix}$ are set to be equal to zero.

This leaves a set of m equations and m unknowns.

These unknowns correspond to the set of basic variables

Theory Behind the Simplex Method

Let the set of basic variables be called $\boldsymbol{x}_B$ and the matrix containing the coefficients of the functional constraints be called $\boldsymbol{A}$ (basis matrix) so that,

$$\boldsymbol{A}\boldsymbol{x}_B = \boldsymbol{b}$$

$$\boldsymbol{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ x_{Bm} \end{bmatrix}$$

The vector $\boldsymbol{x}_B$ is called vector of basic variables.

Theory Behind the Simplex Method

The idea behind each basic feasible solution in the Simplex Algorithm is to eliminate NBV from the set,

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix}$$

and

$$\bar{A} = \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \ldots & \bar{a}_{1m} \\ \bar{a}_{21} & \bar{a}_{22} \ldots & \bar{a}_{2m} \\ \bar{a}_{m1} & \bar{a}_{m2} & \bar{a}_{mm} \end{vmatrix}$$ the basis matrix (a square matrix).

Theory Behind the Simplex Method

From simple matrix algebra (solve for $\boldsymbol{x}_B$) from,

$$\bar{\boldsymbol{A}}\boldsymbol{x}_B = \boldsymbol{b}$$

$$(\bar{\boldsymbol{A}})^{-1}\bar{\boldsymbol{A}}\boldsymbol{x}_B = (\bar{\boldsymbol{A}})^{-1}\boldsymbol{b}$$

$$\boldsymbol{x}_B = (\bar{\boldsymbol{A}})^{-1}\boldsymbol{b}$$

if $\boldsymbol{c}_B$ is the vector of the coefficients of the objective function this brings us to the following value of the objective function:

$$Z = \boldsymbol{c}_B\boldsymbol{x}_B = (\bar{\boldsymbol{A}})^{-1}\boldsymbol{b}$$

Theory Behind the Simplex Method

The original set of equations to start the Simplex Method is,

$$\begin{bmatrix} 1 & -\boldsymbol{c} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{A} & I \end{bmatrix} \begin{bmatrix} Z \\ \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{bmatrix}$$

after each iteration in the Simplex Method,

$$\boldsymbol{x}_B = (\boldsymbol{A})^{-1} \boldsymbol{b}$$

and $Z = \boldsymbol{c}_B \boldsymbol{x}_B = (\boldsymbol{A})^{-1} \boldsymbol{b}$

The RHS of the new set of equations becomes,

Theory Behind the Simplex Method

$$\begin{bmatrix} Z \\ \boldsymbol{x}_B \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1} \\ \mathbf{0} & (\bar{A})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_B(\bar{A})^{-1} b \\ (\bar{A})^{-1} b \end{bmatrix}$$

$$\begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1} \\ \mathbf{0} & (\bar{A})^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\boldsymbol{c} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{A} & I \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1} - \boldsymbol{c} & \boldsymbol{c}_B(\bar{A})^{-1} \\ \boldsymbol{o} & (\bar{A})^{-1}\boldsymbol{A} & (\bar{A})^{-1} \end{bmatrix}$$

After any iteration,

$$\begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1} - \boldsymbol{c} & \boldsymbol{c}_B(\bar{A})^{-1} \\ \boldsymbol{o} & (\bar{A})^{-1}\boldsymbol{A} & (\bar{A})^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_B(\bar{A})^{-1} b \\ (\bar{A})^{-1} b \end{bmatrix}$$

In tableau format this becomes,

Theory of the Simplex Method

Iteration	BV	Z	Original Variables	Slack Variables	RHS
0	Z	1	$-\boldsymbol{c}$	$\mathbf{0}$	0
	$\boldsymbol{x}_B$	$\mathbf{0}$	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
Any	Z	1	$\boldsymbol{c}_B(\bar{A})^{-1}\boldsymbol{A}-\boldsymbol{c}$	$\boldsymbol{c}_B(\bar{A})^{-1}$	$\boldsymbol{c}_B(\bar{A})^{-1}b$
	$\boldsymbol{x}_B$	$\mathbf{0}$	$(\bar{A})^{-1}\boldsymbol{A}$	$(\bar{A})^{-1}$	$(\bar{A})^{-1}b$

Numerical Example

To illustrate the use of the revised simplex method consider the Osaka Bay example:

Maximize $Z = 300x_1 + 500x_2$

subject to: $3x_1 + 2x_2 \leq 180$

$x_1 \leq 40$

$x_2 \leq 60$

$x_1 \geq 0$ and $x_2 \geq 0$

Note: let x_1 and x_2 be the no. "Fuji" and "Haneda" vessels

note that: $\boldsymbol{c} = \left[\begin{bmatrix} 300 & 500 \end{bmatrix}\right]$ coefficients of real variables

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix}$, $\boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and matrix $\boldsymbol{A}$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Theory Behind the Simplex Method

Addition of slack variables to the problem yields:

$$\boldsymbol{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$ where $\boldsymbol{x}_s$ is a vector of slack variables

Executing the procedure for the Simplex Method

Iteration 0:

$$\boldsymbol{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \ (\bar{\boldsymbol{A}})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix}$$

also known,

$\boldsymbol{c}_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and hence $Z = \boldsymbol{c}_B \boldsymbol{x}_B = (\boldsymbol{A})^{-1} \boldsymbol{b}$ or

$$Z = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix} = 0$$

Iteration 1: (refer to 2nd tableau in Simplex)

Note: substitute values for $\bar{\boldsymbol{A}}$ using columns for x_3 , x_4 and x_2 in the original A matrix.

$$\boldsymbol{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_2 \end{bmatrix}, \bar{\boldsymbol{A}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{\boldsymbol{A}}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and

$$\begin{bmatrix} x_3 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \\ 60 \end{bmatrix}$$

also known,

$\boldsymbol{c}_B = \begin{bmatrix} 0 & 0 & 500 \end{bmatrix}$ and hence $Z = \boldsymbol{c}_B \boldsymbol{x}_B = (\boldsymbol{A})^{-1} \boldsymbol{b}$ or

$$Z = \begin{bmatrix} 0 & 0 & 500 \end{bmatrix} \begin{bmatrix} 60 \\ 40 \\ 60 \end{bmatrix} = 30000$$

Iteration 2: (refer to 3rd tableau in Simplex)

Note: substitute values for $\bar{A}$ using columns for x_1, x_4 and x_2 in the original A matrix.

$$\boldsymbol{x}_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix}, \bar{\boldsymbol{A}} = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \bar{\boldsymbol{A}}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$ and

$$\begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 60 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 60 \end{bmatrix}$$

also known,

$\boldsymbol{c}_B = [300 \;\; 0 \;\; 500]$ and hence $Z = \boldsymbol{c}_B \boldsymbol{x}_B = (A)^{-1}\boldsymbol{b}$ or

$$Z = [300 \;\; 0 \;\; 500]\begin{bmatrix} 20 \\ 20 \\ 60 \end{bmatrix} = 36000$$ **Optimal Solution**

Linear Programming Programs

Several computer programs are available to solve LP problems:

- LINDO - Linear INteractive Discrete Optimizer
- GAMS - also solves non linear problems
- MINUS
- Matlab Toolbox - Optimization toolbox (from Mathworks)
- QSB - LP, DP, IP and other routines available (good for students)