Mathematical Programming (LP) and Excel Solver

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Recall - Linear Programming

General Formulation

Maximize $\sum_{j=1}^{n} c_j x_j$

subject to: $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ for $i = 1, 2, \ldots, m$

$x_j \ge 0$ for $j = 1, 2, \ldots, n$

Linear Programming

$$\sum_{j=1}^{n} c_j x_j$$ Objective Function (OF)

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$ Functional Constraints (*m* of them)

$x_j \ge 0$ Nonnegativity Conditions (*n* of these)

x_j are decision variables to be optimized (min or max)

c_j are costs associated with each decision variable

Linear Programming

a_{ij} are the coefficients of the functional constraints

b_i are the amounts of the resources available (RHS)

LP Example (Construction)

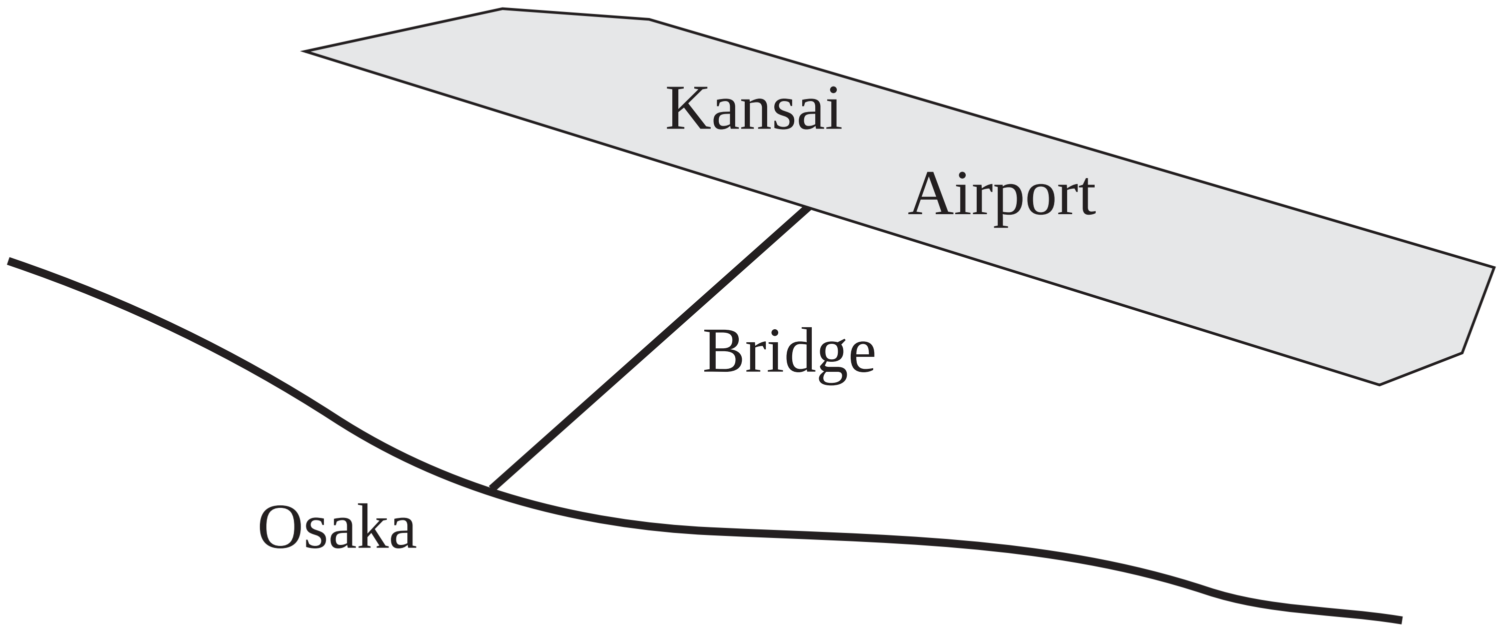
During the construction of an off-shore airport in Japan the main contractor used two types of cargo barges to transport materials from a fill collection site to the artificial island built to accommodate the airport.

The types of cargo vessels have different cargo capacities and crew member requirements as shown in the table:

Vessel Type	Capacity (m-ton)	Crew required	Number available
Fuji	300	3	40
Haneda	500	2	60

Osaka Bay Model

According to company records there are 180 crew members in the payroll and all crew members are trained to either manage the “Haneda” or “Fuji” vessels.



Osaka Bay Model

Mathematical Formulation

Maximize $Z = 300x_1 + 500x_2$

subject to: $3x_1 + 2x_2 \leq 180$

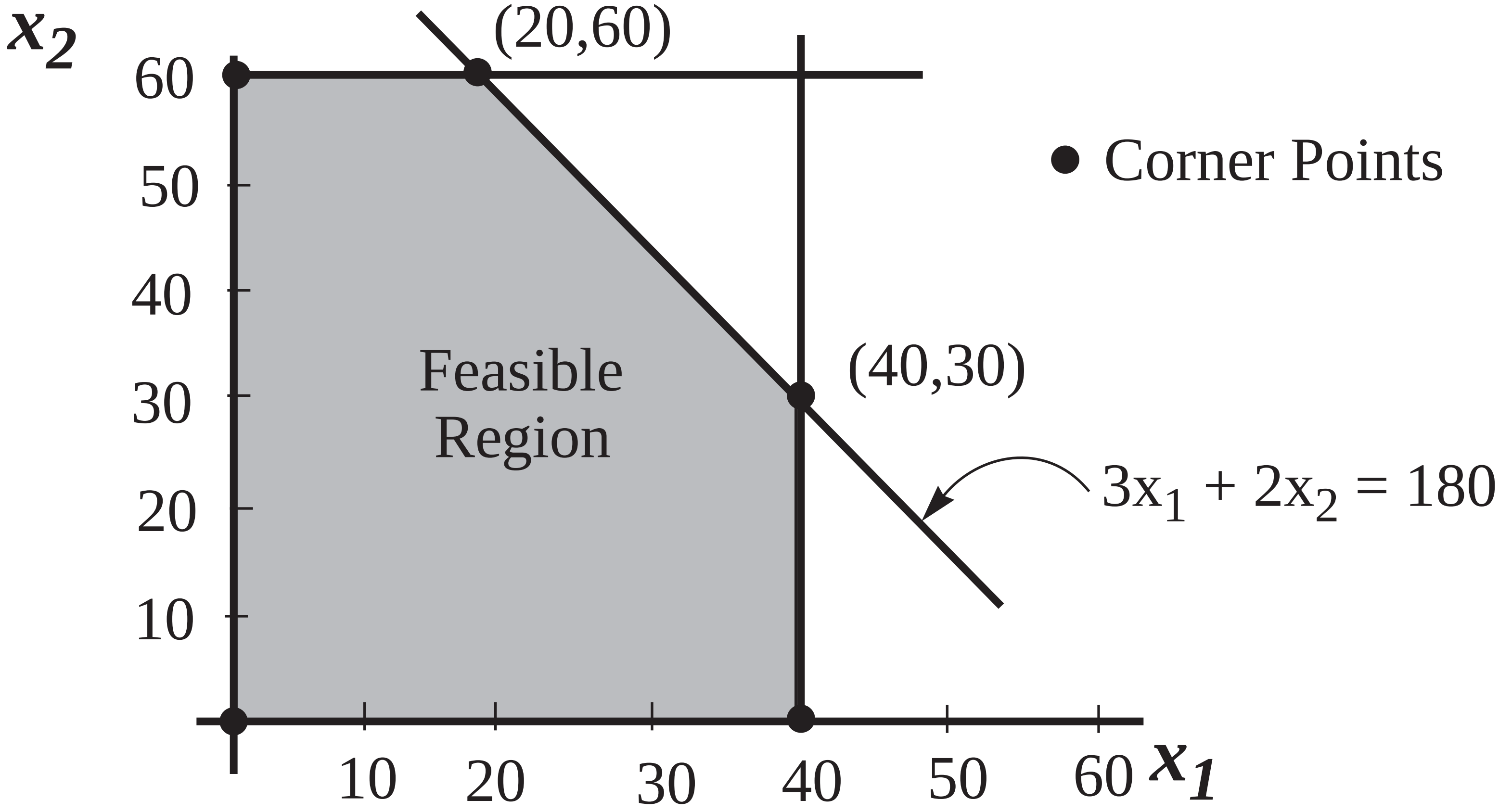
$x_1 \leq 40$

$x_2 \leq 60$

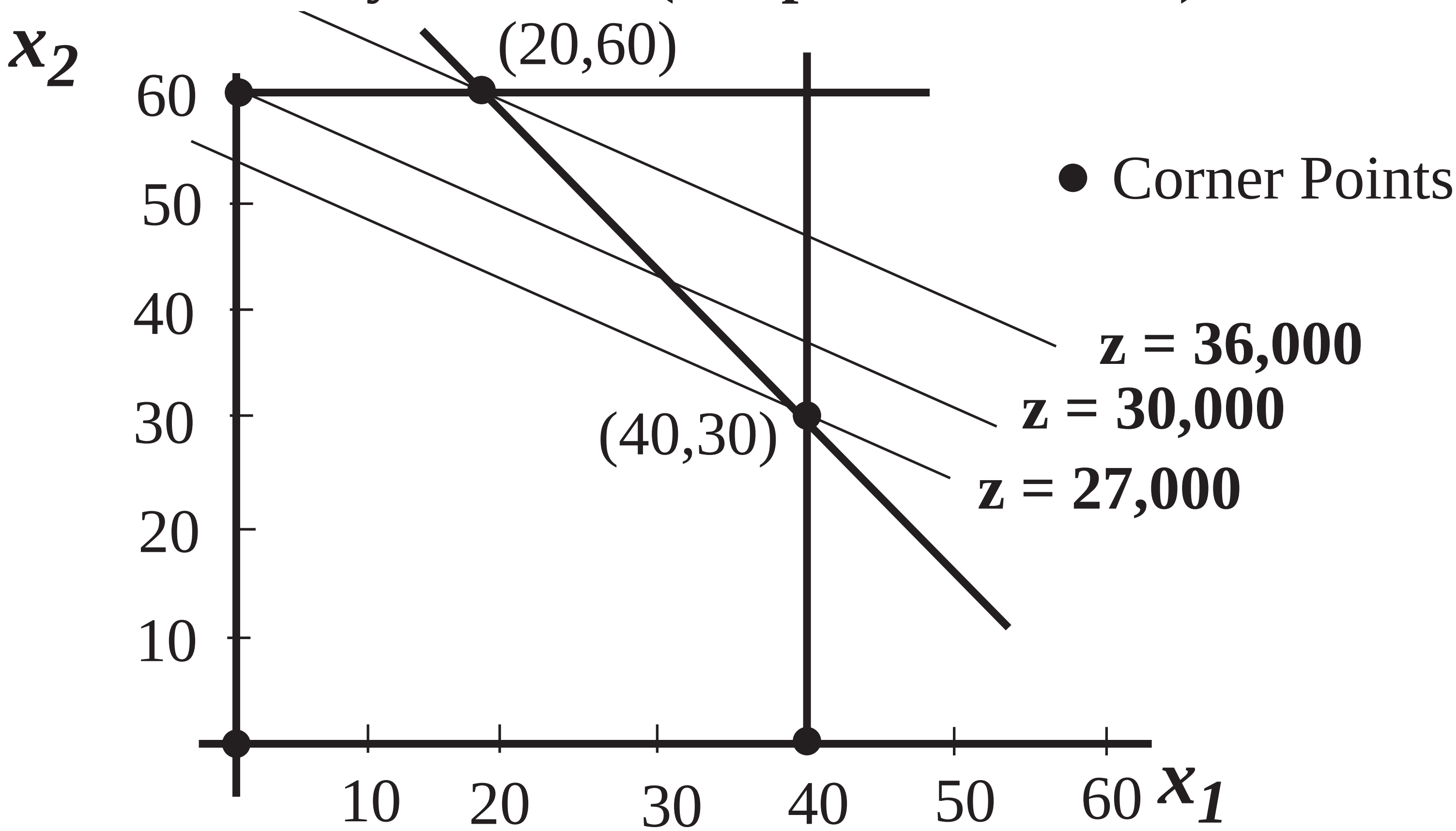
$x_1 \geq 0$ and $x_2 \geq 0$

Note: let x_1 and x_2 be the no. "Fuji" and "Haneda" vessels

Osaka Bay Problem (Graphical Solution)



Osaka Bay Problem (Graphical Solution)



Note: Optimal Solution (x_1, x_2) = (20,60) vessels

Solution Using Excel Solver

- Solver is a Generalized Reduced Gradient (GRG2) nonlinear optimization code
- Developed by Leon Lasdon (UT Austin) and Allan Waren (Cleveland State University)
- Optimization in Excel uses the Solver add-in.
- Solver allows for one function to be minimized, maximized, or set equal to a specific value.
- Convergence criteria (convergence), integer constraint criteria (tolerance), and are accessible through the OPTIONS button.

Excel Solver

- Excel can solve simultaneous linear equations using matrix functions
- Excel can solve one nonlinear equation using Goal Seek or Solver
- Excel does not have direct capabilities of solving n multiple nonlinear equations in n unknowns, but sometimes the problem can be rearranged as a minimization function

Osaka Bay Problem in Excel

Optimization Problem for Osaka Bay

Decision Variables		
x1	20	Number of Ships Type 1
x2	60	Number of Ships Type 2

Objective Function	
300 x1 + 500 x2	36000

Objective function
Stuff to be solved

Constraint Equations			
	Formula		
3 x1 + 2 x2 <= 180	180	<=	180
x1 <= 40	20	<=	40
x2 <= 60	60	<=	60
x1 >= 0	20	>=	0
x2 >= 0	60	>=	0

Osaka Bay Problem in Excel

Optimization Problem for Osaka Bay

Decision variables (what your control)

Decision Variables			
x1	20		Number of Ships Type 1
x2	60		Number of Ships Type 2

Objective Function	
300 x1 + 500 x2	36000

Constraint Equations			
	Formula		
3 x1 + 2 x2 <= 180	180	<=	180
x1 <= 40	20	<=	40
x2 <= 60	60	<=	60
x1 >= 0	20	>=	0
x2 >= 0	60	>=	0

Osaka Bay Problem in Excel

Optimization Problem for Osaka Bay

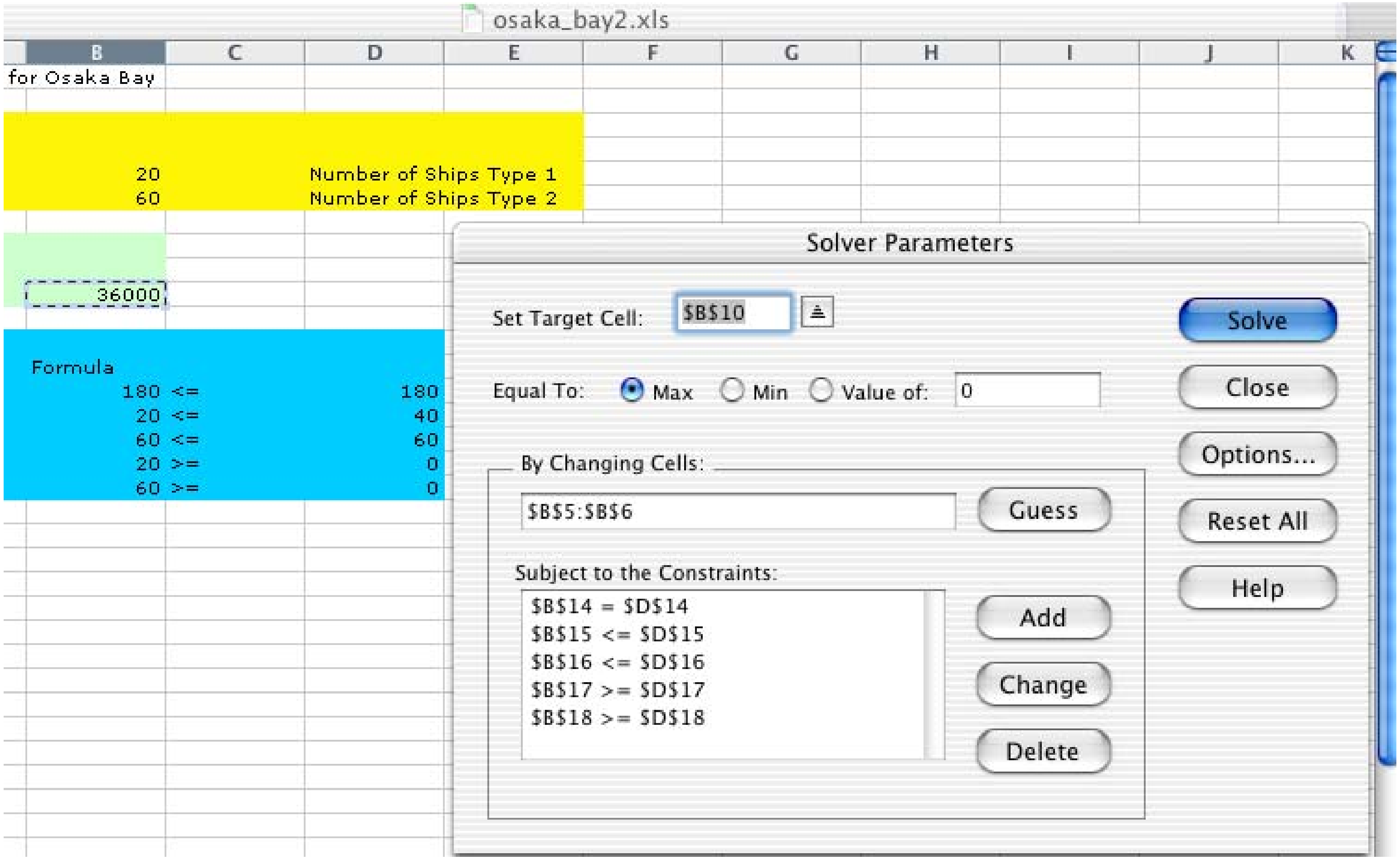
Decision Variables		
x1	20	Number of Ships Type 1
x2	60	Number of Ships Type 2

Objective Function	
300 x1 + 500 x2	36000

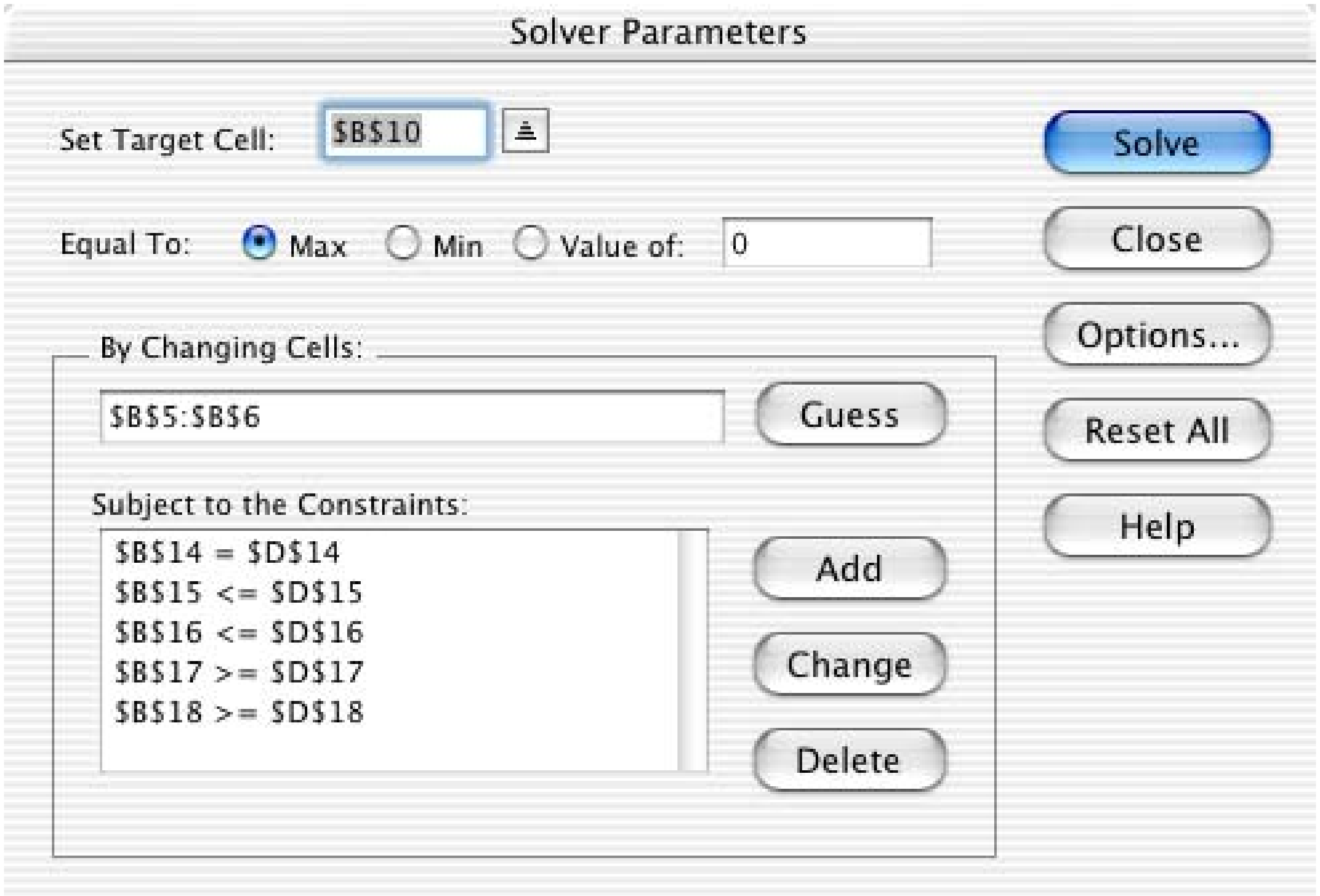
Constraint equations (limits to the problem)

Constraint Equations			
	Formula		
3 x1 + 2 x2 <= 180	180	<=	180
x1 <= 40	20	<=	40
x2 <= 60	60	<=	60
x1 >= 0	20	>=	0
x2 >= 0	60	>=	0

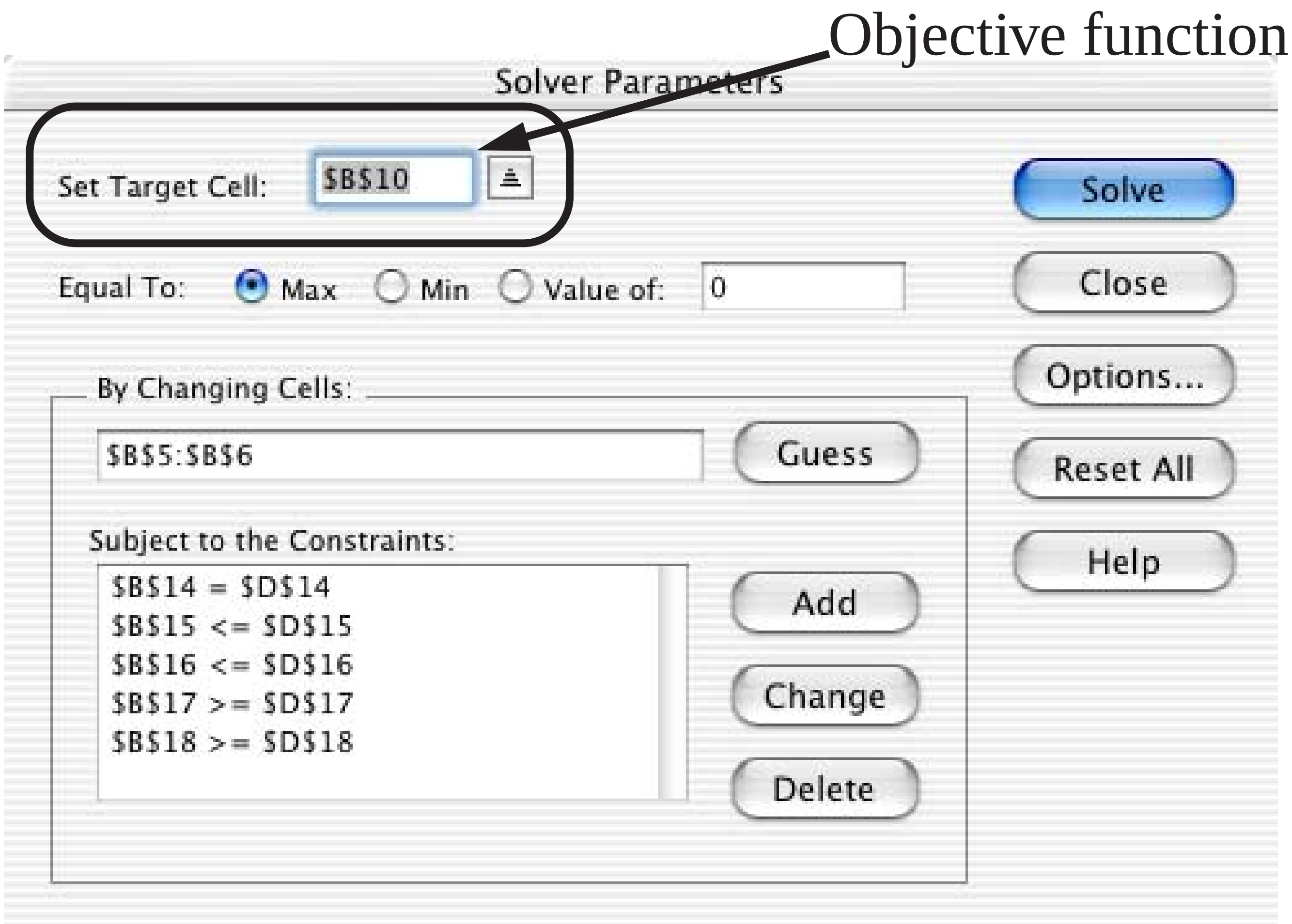
Solver Panel in Excel



Solver Panel in Excel

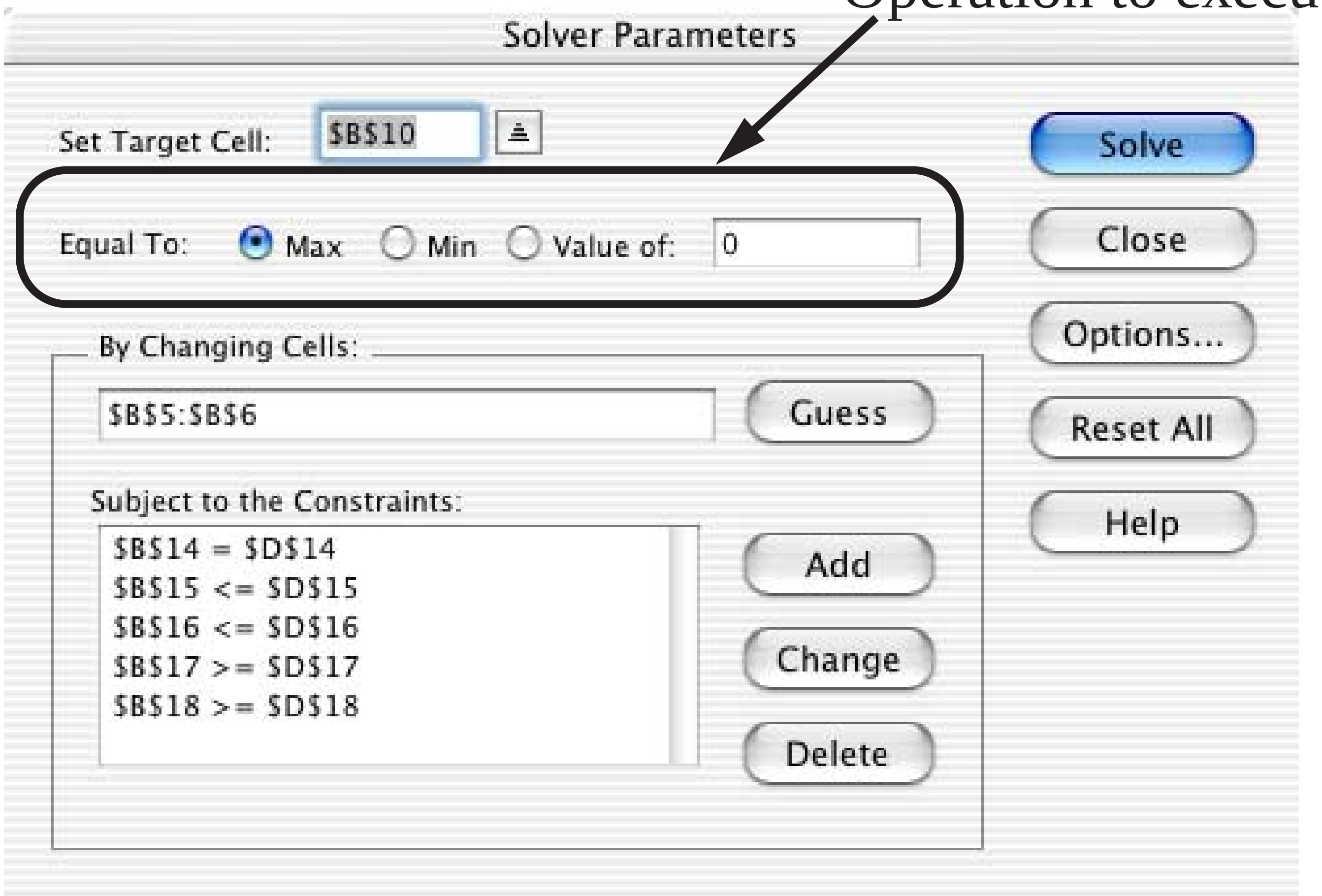


Solver Panel in Excel



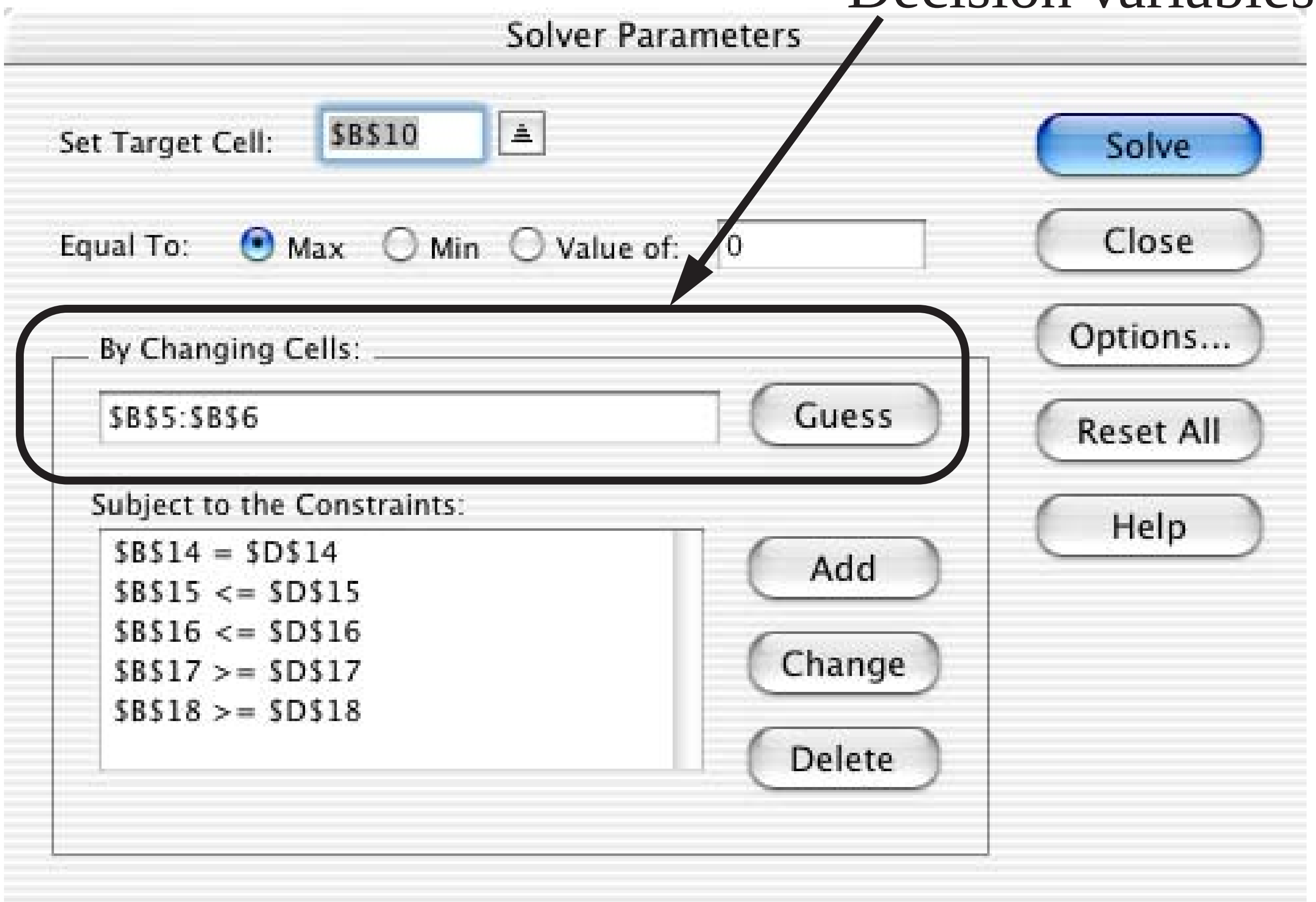
Solver Panel in Excel

Operation to execute



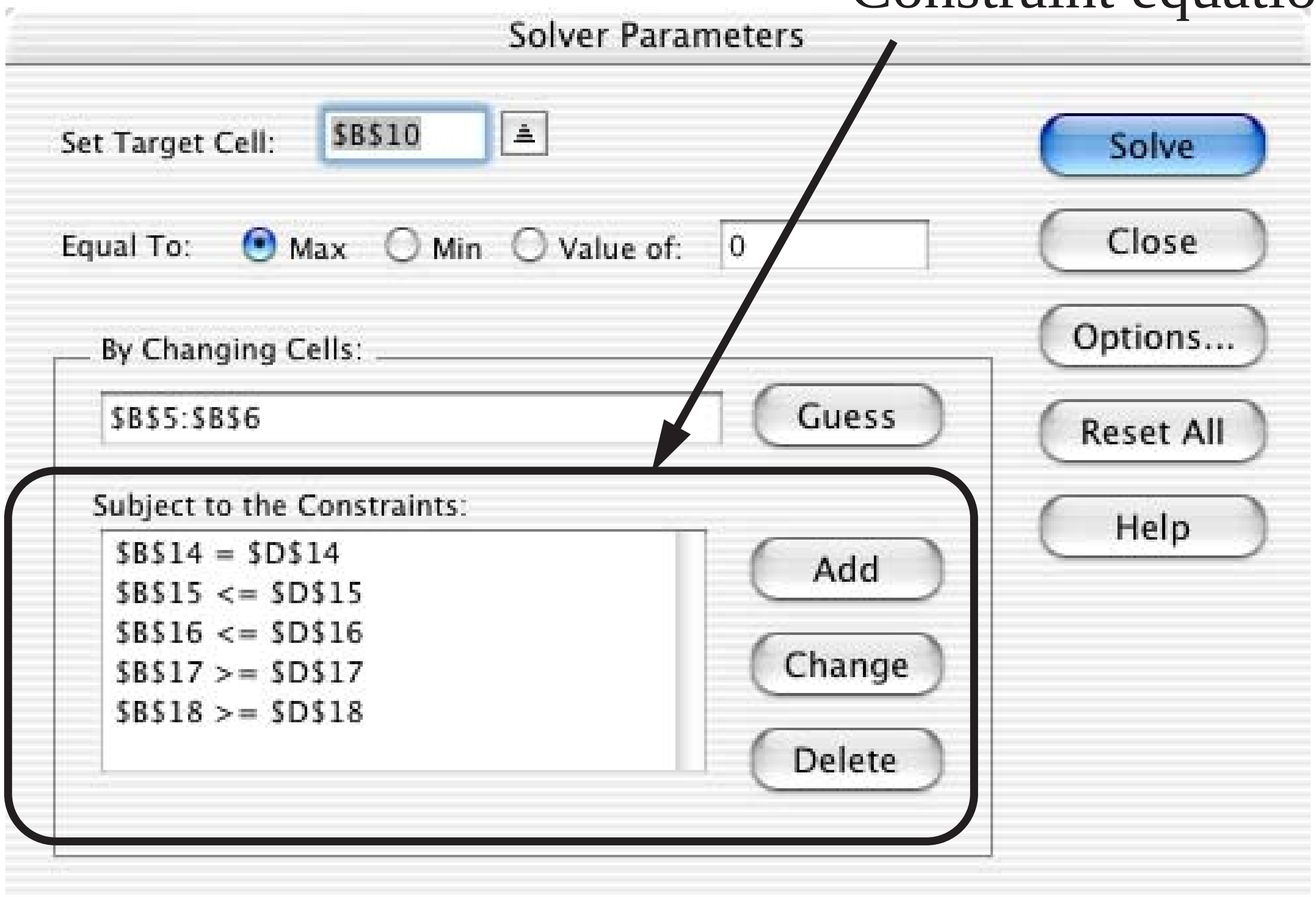
Solver Panel in Excel

Decision variables

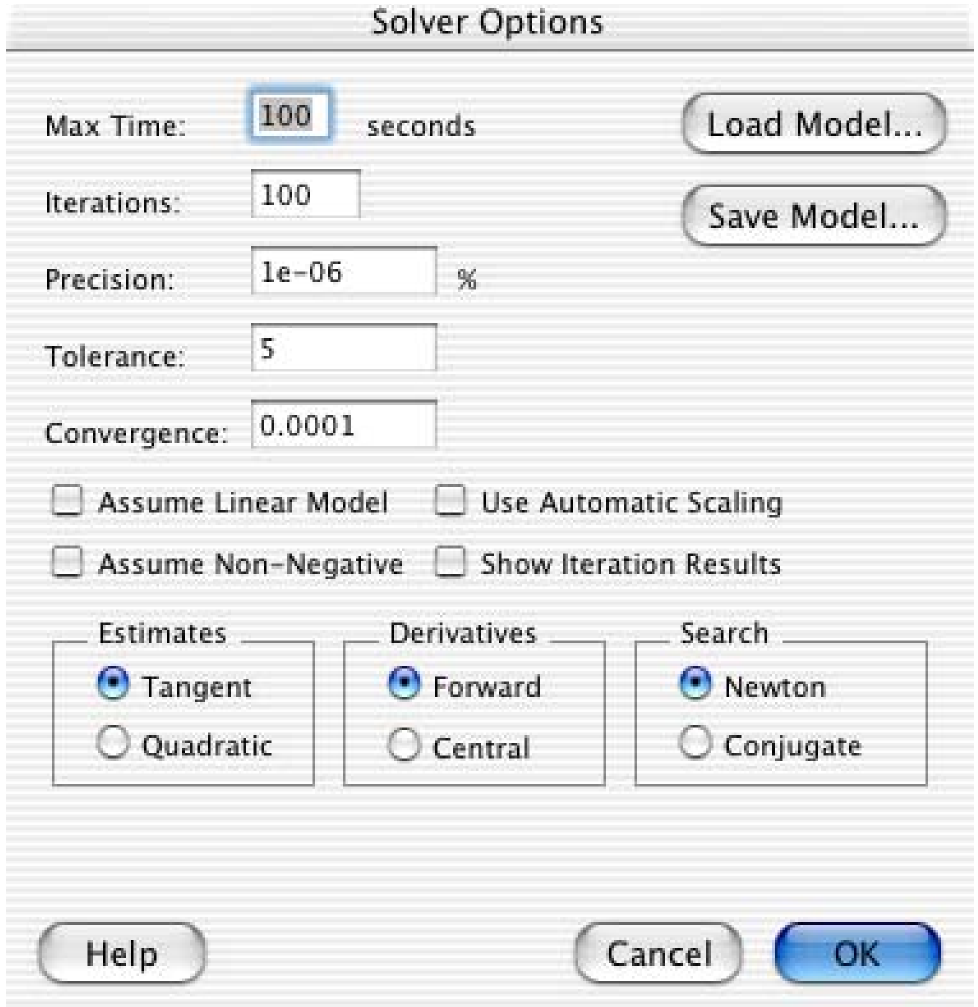


Solver Panel in Excel

Constraint equations



Solver Options Panel Excel



Excel Solver Limits Report

- Provides information about the limits of decision variables

osaka_bay2.xls

Microsoft Excel 10.1 Limits Report
Worksheet: [Workbook1]Sheet1
Report Created: 3/10/2003 5:04:26 AM

Cell	Target Name	Value
B10	300 x1 + 500 x2	36000

Cell	Adjustable Name	Value	Lower Limit	Target Result	Upper Limit	Target Result
B5	x1	20	0	30000	20	36000
B6	x2	60	1.33227E-15	6000	60	36000

Excel Solver Sensitivity Report

- Provides information about shadow prices of decision variables

osaka_bay2.xls

Microsoft Excel 10.1 Sensitivity Report
Worksheet: [osaka_bay2.xls]Sheet1
Report Created: 3/10/2003 5:47:49 AM

Adjustable Cells

Cell	Name	Final Value	Reduced Gradient
B5	x1	20	0
B6	x2	60	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
B18	x2 >= 0 Formula	60	0
B15	x1 <= 40 Formula	20	0
B16	x2 <= 60 Formula	60	300
B17	x1 >= 0 Formula	20	0
B14	3 x1 + 2 x2 <= 180 Formula	180	100

Unconstrained Optimization Problems

- Common in engineering applications
- Can be solved using Excel solver as well
- The idea is to write an equation (linear or nonlinear) and then use solver to iterate the variable (or variables) to solve the problem

Simple One Dimensional Unconstrained Optimization

- Given the quadratic equation

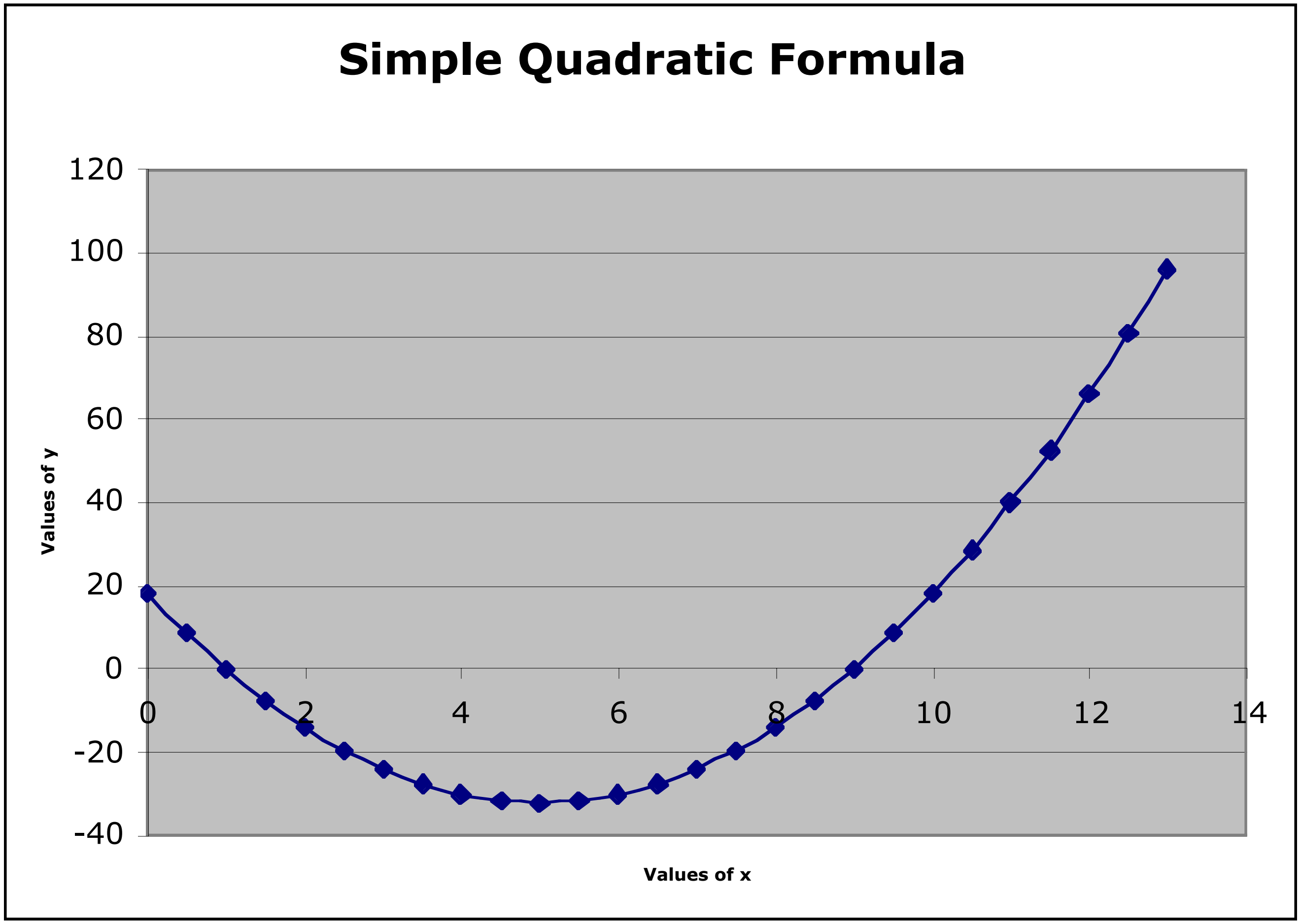
$$y = 2x^2 - 20x + 18$$

- Find the minima of the equation for all values of x

Solution:

- Lets try the Excel Solver

Plot of Equation to be Solved



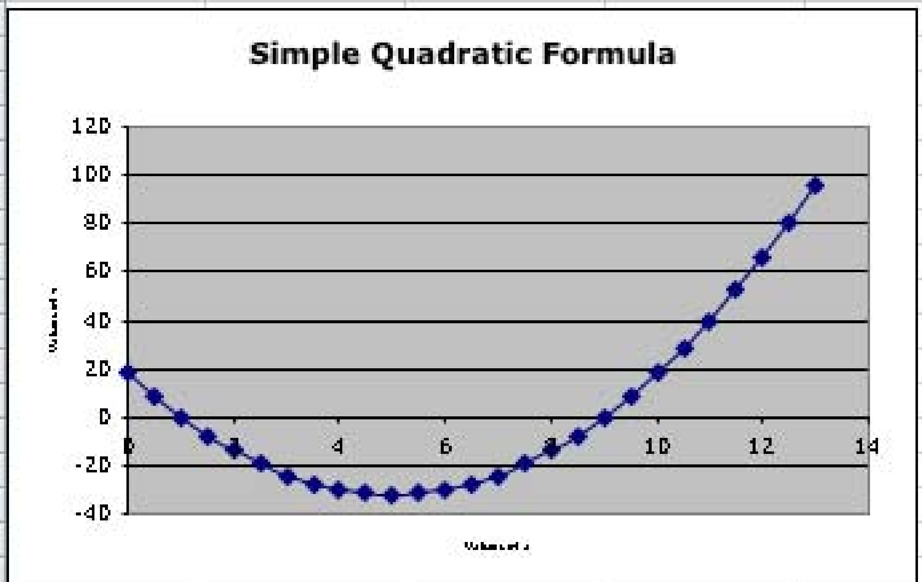
Excel Solver Procedure

Virginia Tech

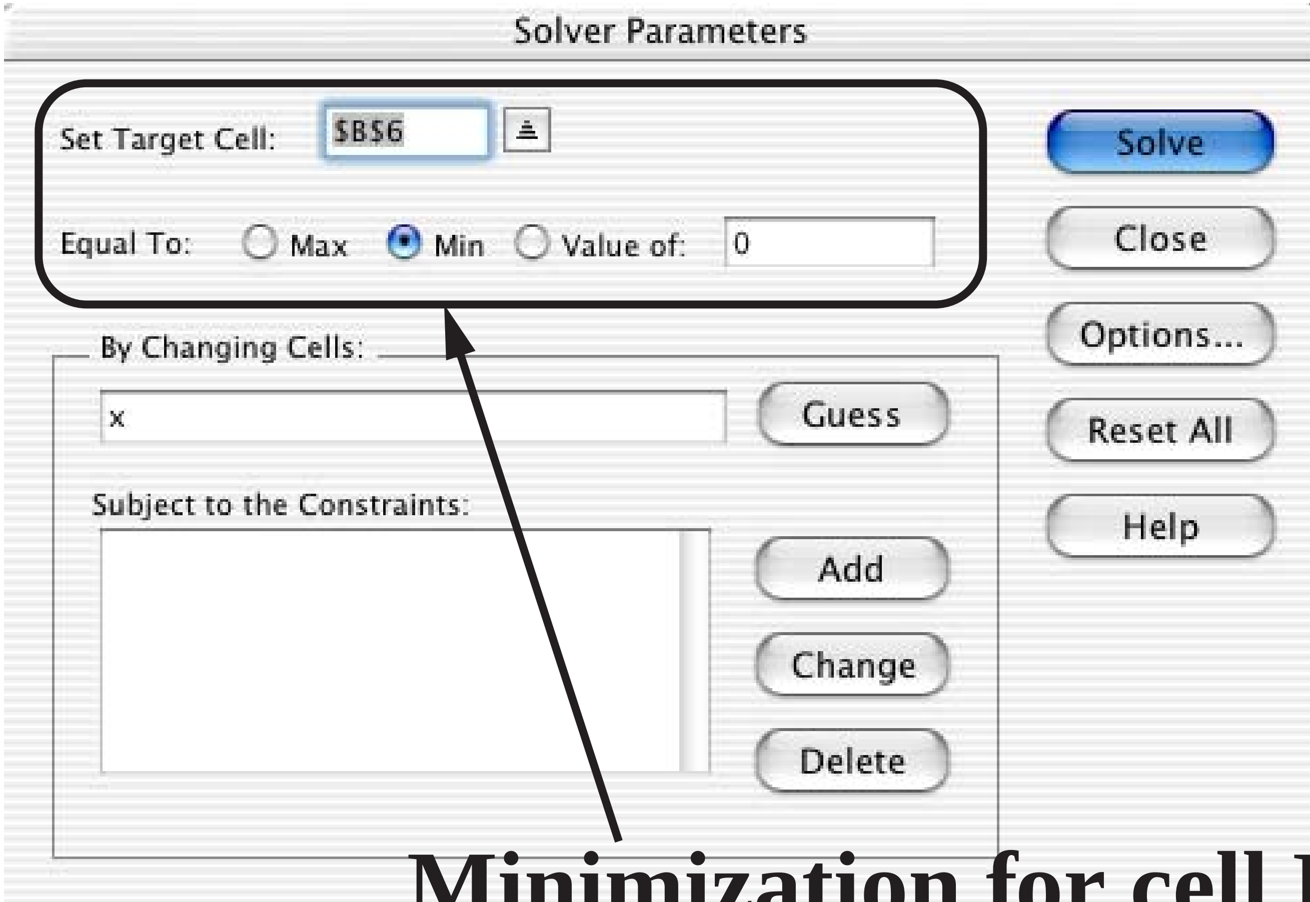
minima_example.xls

Guess value of x

	A	B	C	D	E	F	G
1	Finding the Minima of a function						
2							
3	Function	y=2*x^2 - 20 * x + 18					
4							
5	Guess	5			Values of x	y	
6	Equation	-32			0	18	
7					0.5	8.5	
8					1	0	
9					1.5	-7.5	
10					2	-14	
11					2.5	-19.5	
12					3	-24	
13					3.5	-27.5	
14					4	-30	
15					4.5	-31.5	
16					5	-32	
17					5.5	-31.5	
18					6	-30	
19					6.5	-27.5	
20					7	-24	
21					7.5	-19.5	
22					8	-14	
23					8.5	-7.5	
24					9	0	
25					9.5	8.5	
26					10	18	
27					10.5	28.5	

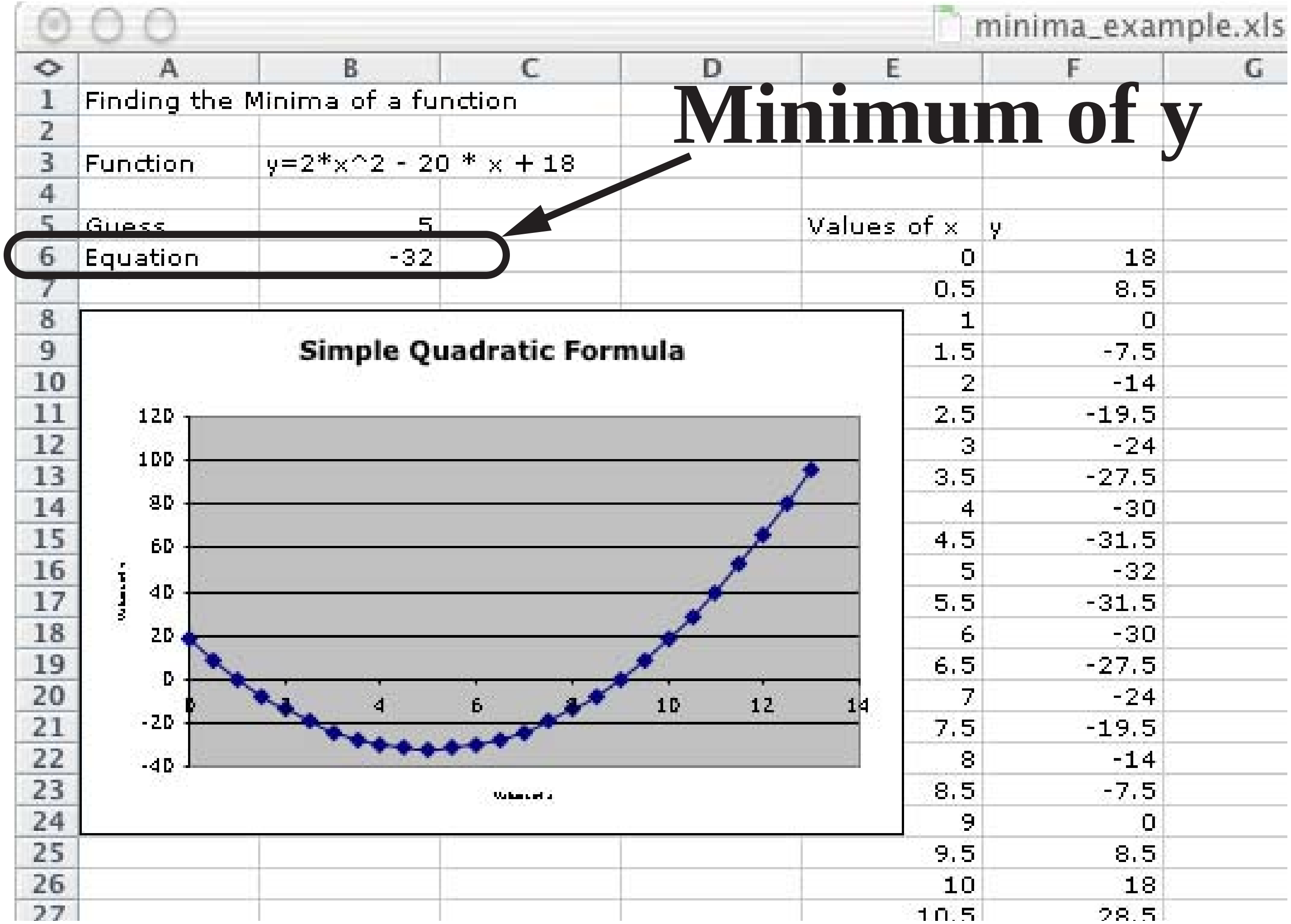


Excel Solver Panel



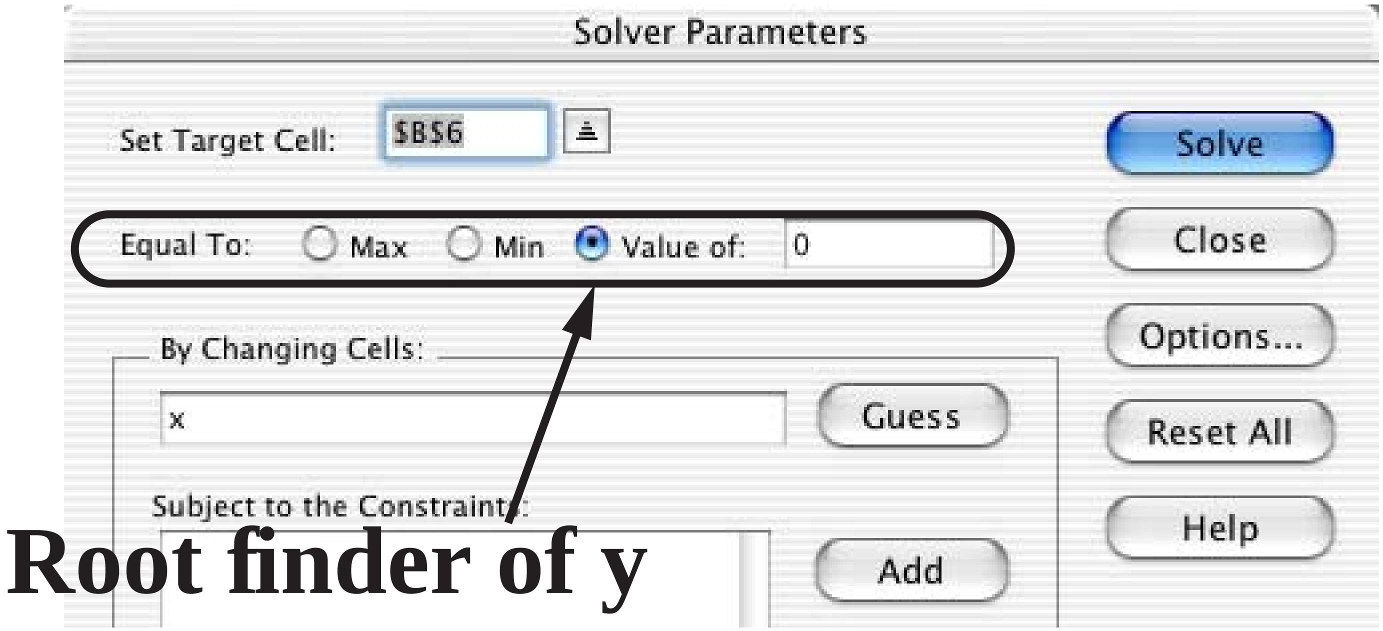
Minimization for cell B6

Excel Solver Procedure



Finding the Roots of y Using Excel Solver

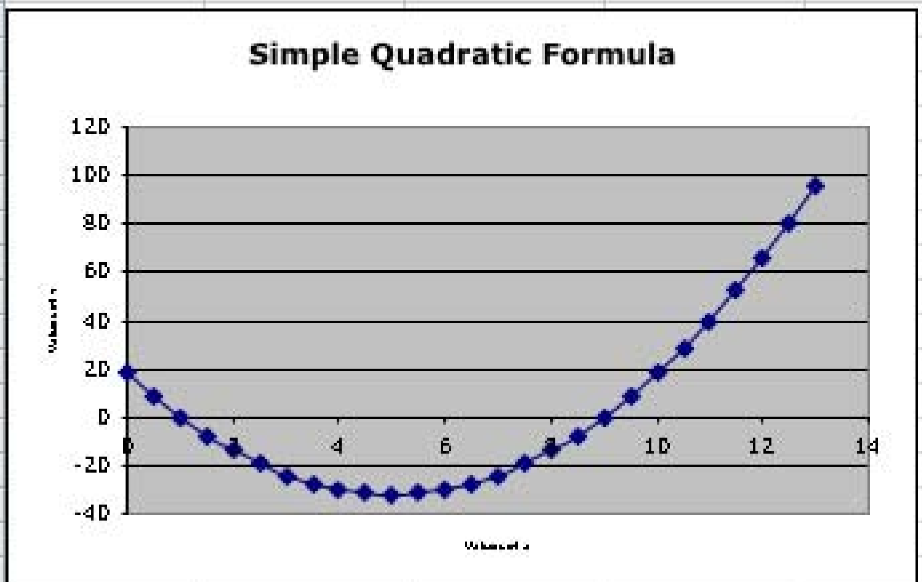
- Easily change the minimimzation problem into a root finder by changing the character of the operation in Excel Solver



Root Finder for y

minima_example.xls

	A	B	C	D	E	F	G
1	Finding the Minima of a function						
2							
3	Function	y=2*x^2 - 20 * x + 18					
4							
5	Guess	9			Values of x	y	
6	Equation	-7.031E-08			0	18	
7					0.5	8.5	
8					1	0	
9					1.5	-7.5	
10					2	-14	
11					2.5	-19.5	
12					3	-24	
13					3.5	-27.5	
14					4	-30	
15					4.5	-31.5	
16					5	-32	
17					5.5	-31.5	
18					6	-30	
19					6.5	-27.5	
20					7	-24	
21					7.5	-19.5	
22					8	-14	
23					8.5	-7.5	
24					9	0	
25					9.5	8.5	
26					10	18	



Example for Class Practice

- Minimization example (mixing problem)
- Airline fleet assignment problem

Minimization LP Example

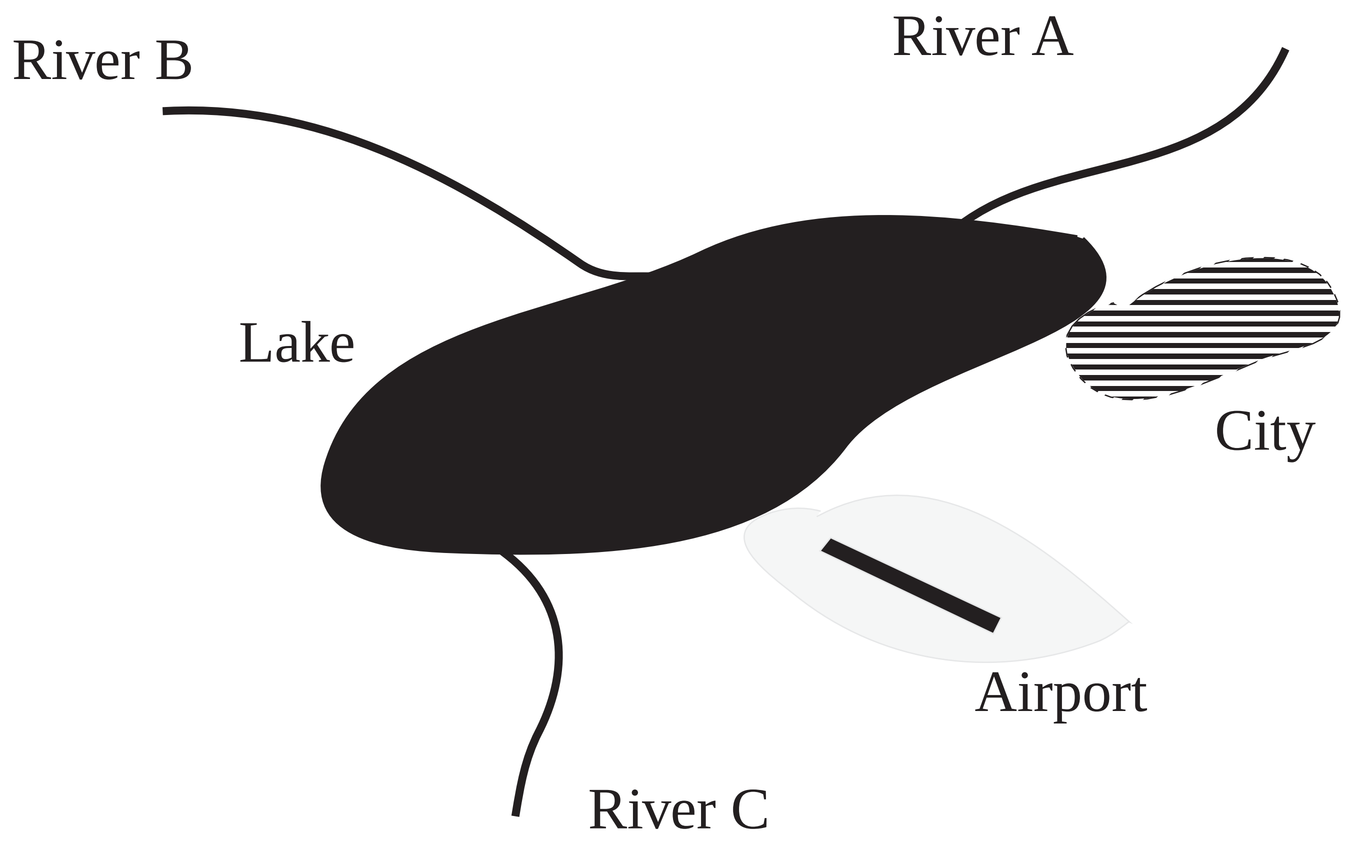
A construction site requires a minimum of 10,000 cu. meters of sand and gravel mixture. The mixture must contain no less than 5,000 cu. meters of sand and no more than 6,000 cu. meters of gravel.

Materials may be obtained from two sites: 30% of sand and 70% gravel from site 1 at a delivery cost of $5.00 per cu. meter and 60% sand and 40% gravel from site 2 at a delivery cost of $7.00 per cu. meter.

a) Formulate the problem as a Linear Programming problem

b) Solve using Excel Solver

Application to Water Pollution



Water Pollution Management

The following are pollution loadings due to five sources:

Note: Pollution removal schemes vary in cost dramatically.

Source	Pollution Loading (kg/yr)	Unit Cost of Removal ($/kg)
River A	18,868	1.2
River B	20,816	1.0
River C	37,072	0.8
Airport	28,200	2.2
City	12,650	123.3

Water Pollution Management

It is desired to reduce the total pollution discharge to the lake to 70,000 kg/yr. Therefore the target pollution reduction is 117,606-70,000 = 47,606 kg/yr.

Solution:

Let x_1, x_2, x_3, x_4, x_5 be the pollution reduction values expected in (kg/yr). The costs of unit reduction of pollution are given in the previous table.

The total pollution reduction from all sources should be at least equal to the target reduction of 47,606 kg.

LP Applications - Water Pollution Management

The reductions for each source cannot be greater than the present pollution levels. Mathematically,

$x_1 \leq 18868$ constraint for River A

$x_2 \leq 20816$ constraint for River B

$x_3 \leq 37072$ constraint for River C

$x_4 \leq 28200$ airport constraint

$x_5 \leq 12650$ city constraint

Water Pollution Management

The reductions at each source should also be non negative.

Using this information we characterize the problem as follows:

Min $z = 1.2x_1 + 1.0x_2 + 0.8x_3 + 2.2x_4 + 123.3x_5$

s.t. $x_1 + x_2 + x_3 + x_4 + x_5 \geq 47606$

$x_1 \leq 18868$

$x_2 \leq 20816$

$x_3 \leq 37072$

$x_4 \leq 28200$ and $x_5 \leq 12650$

Water Resource Management

Rewrite the objective function as follows:

Max $-z + 1.2x_1 + 1.0x_2 + 0.8x_3 + 2.2x_4 + 123.3x_5 + Mx_{12}$

st. $x_1 + x_2 + x_3 + x_4 + x_5 - x_6 + x_{12} = 47606$

$$x_1 + x_7 = 18868$$

$$x_2 + x_8 = 20816$$

$$x_3 + x_9 = 37072$$

$$x_4 + x_{10} = 28200$$

$$x_5 + x_{11} = 12650$$

Solution in Matlab (Input File)

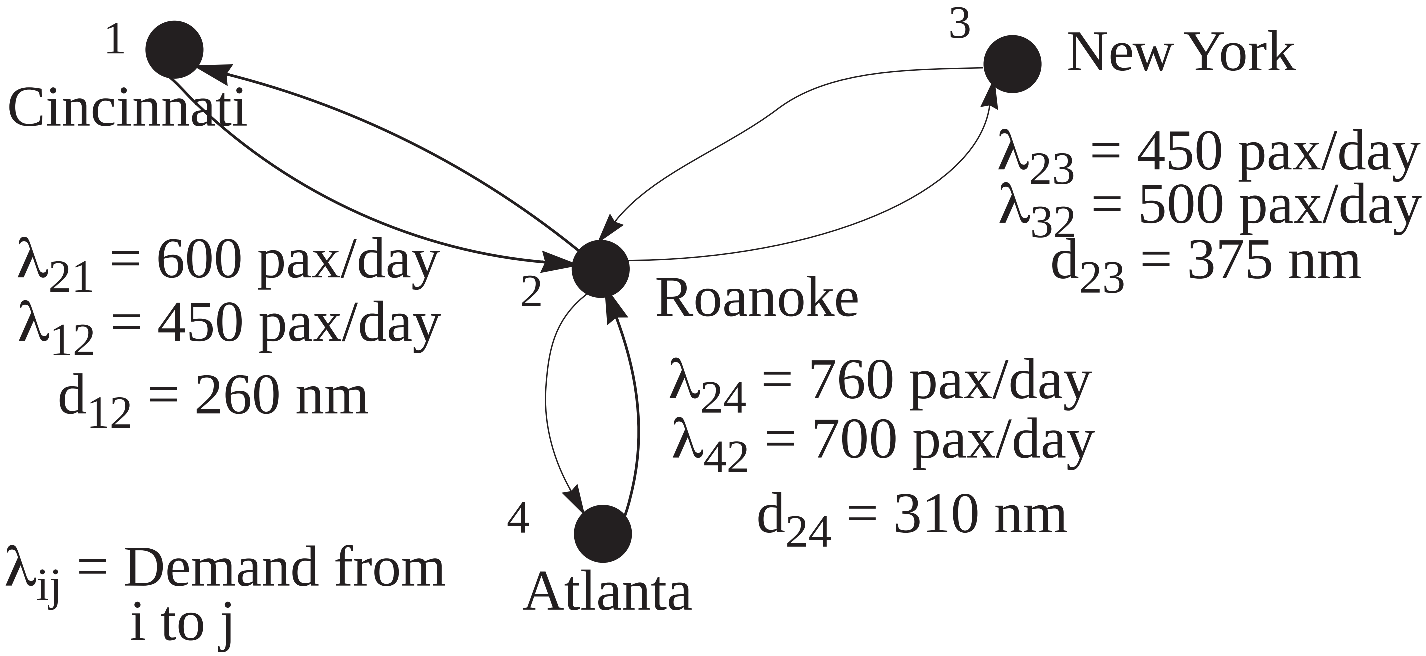
```
% Example: Enter the data:
  minmax=1;% minimizing problem
  a=[1 1 1 1 1 -1 0 0 0 0 0 1
     1 0 0 0 0 0 1 0 0 0 0 0
     0 1 0 0 0 0 0 1 0 0 0 0
     0 0 1 0 0 0 0 0 1 0 0 0
     0 0 0 1 0 0 0 0 0 1 0 0
     0 0 0 0 1 0 0 0 0 0 1 0]
  b=[47606 18868 20816 37072 28200 12650]'
  c=[-1.2 -1. -.8 -2.2 -123.3 0 0 0 0 0 0 -999]
  bas=[12 7 8 9 10 11]
```

Try it in Excel Solver!

Airline Scheduling Problem

A small airline would like to use mathematical programming to schedule its flights to maximize profit.

The following map shows the city pairs to be operated.

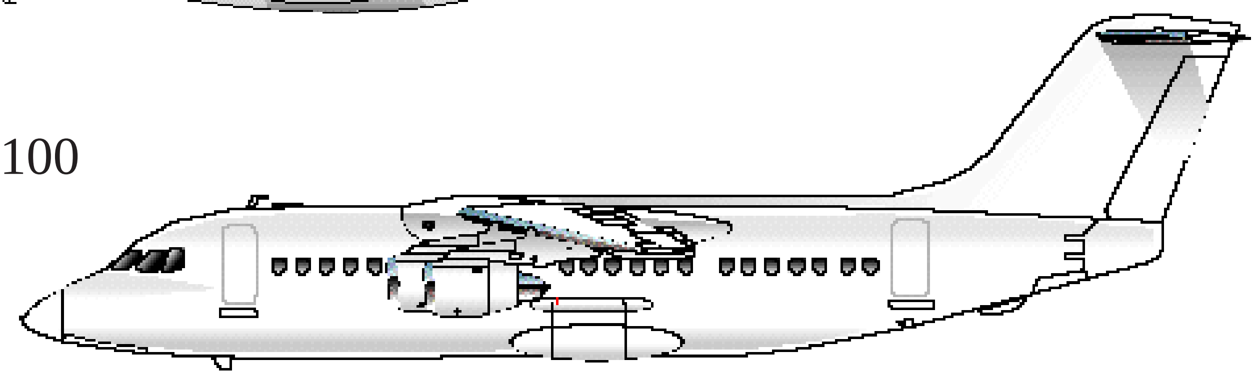


Airline Scheduling Problem

The airline has decided to purchase two types of aircraft to satisfy its needs: 1) the Embraer 145, a 45-seat regional jet, and 2) the Avro RJ-100, a four-engine 100 seater aircraft (see the following figure).

EMB-145

Avro RJ-100



Aircraft Characteristics

The table has pertinent characteristics of these aircraft

Aircraft	**EMB-145**	**Avro RJ-100**
Seating capacity - n_k	50	100
Block speed (knots) - v_k	400	425
Operating cost ($/hr) - c_k	1,850	3,800
Maximum aircraft utilization (hr/day)[a] - U_k	13.0	12.0

a. The aircraft utilization represents the maximum number of hours an aircraft is in actual use with the engines running (in airline parlance this is the sum of all daily block times). Turnaround times at the airport are not part of the utilization variable as defined here.

Nomenclature

Define the following sets of decision variables:

No. of acft. of type k in fleet = A_k

No. flights assigned from i to j using aircraft of type k = N_{ijk}

Minimum flight frequency between i and j = $(N_{ij})_{min}$

Based on expected load factors, the tentative fares between origin and destination pairs are indicated in the following table.

City pair designator	**Origin-Destination**	**Average one-way fare ($/seat)**
ROA-CVG	Roanoke to Cincinnati	175.00
ROA-LGA	Roanoke to La Guardia	230.00
ROA-ATL	Roanoke to Atlanta	200.00

Problem # 1 Formulation

1) Write a mathematical programming formulation to solve the ASP-1 Problem with the following constraints:

Maximize **Profit**

subject to:

- aircraft availability constraint
- demand fulfillment constraint
- minimum frequency constraint

Problem # 2 ASP-1 Solution

1) Solve problem ASP-1 under the following numerical assumptions:

a) Maximize profit solving for the fleet size and frequency assignment without a minimum frequency constraint. Find the number of aircraft of each type and the number of flights between each origin-destination pair to satisfy the two basic constraints (demand and supply constraints).

b) Repeat part (a) if the minimum number of flights in the arc ROA-ATL is 8 per day (8 more from ATL-ROA) to establish a shuttle system between these city pairs.

Vehicle Scheduling Problem

Formulation of the problem.

Maximize **Profit**

subject to: (possible types of constraints)

a) aircraft availability constraint

b) demand fulfillment constraint

c) Minimum frequency constraint

d) Landing restriction constraint

Vehicle Scheduling Problem

Profit Function

P = Revenue - Cost

Revenue Function

$$\text{Revenue} = \sum_{(i,j)} \lambda_{ij} f_{ij}$$

where: λ_{ij} is the demand from i to j (daily demand)

f_{ij} is the average fare flying from i to j

Vehicle Scheduling Problem

Cost function

let N_{ijk} be the flight frequency from i to j using aircraft type k

let C_{ijk} be the total cost per flight from i to j using aircraft k

$$\text{Cost} = \sum_{(i,j)} \sum_{k} N_{ijk} C_{ijk}$$

then the profit function becomes,

Profit = $\sum_{i,j} \lambda_{ij} f_{ij} - \sum_{i,j} \sum_{k} N_{ijk} C_{ijk}$

Vehicle Scheduling Problem

Demand fulfillment constraint

Supply of seats offered > Demand for service

$\sum_k n_k N_{ijk} \geq \lambda_{ij}$ for all (i, j) city pairs or alternatively

$\sum_k (lf) n_k N_{ijk} \geq \lambda_{ij}$ for all (i, j) city pairs

lf is the load factor desired in the operation (0.8-0.85)

Note: airlines actually overbook flights so they usually factor a target load factor in their schedules to account for some slack

Vehicle Scheduling Problem

Aircraft availability constraint

(block time) (no. of flights) < (utilization)(no. of aircraft)

$$\sum_{(i,j)} t_{ijk} N_{ijk} \leq U_k A_k$$

one constraint equation for every k aircraft type

Vehicle Scheduling Problem

Minimum frequency constraint

No. of flights between i and j > Minimum number of desired flights

$$\sum_{k} N_{ijk} \geq (N_{ij})_{min} \text{ for all } (i, j) \text{ city pairs}$$

Note: Airlines use this strategy to gain market share in highly traveled markets

Vehicle Scheduling Problem

Maximize Profit = $$\sum_{i,j} \lambda_{ij} f_{ij} - \sum_{i,j} \sum_{k} N_{ijk} C_{ijk}$$

subject to

$$\sum_{k} n_k N_{ijk} \geq \lambda_{ij}$$ for all (i, j) city pairs

$$\sum_{(i,j)} t_{ijk} N_{ijk} \leq U_k A_k$$ for every k aircraft type

$$\sum_{k} N_{ijk} \geq (N_{ij})_{min}$$ for all (i, j) city pairs