Analysis of Air Transportation Systems

Mathematical Programming Applications

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Resource Allocation

Principles of **Mathematical Programming**

Mathematical programming is a general technique to solve resource allocation problems using optimization. Types of problems:

- Linear programming
- Integer programming
- Dynamic programming
- Decision analysis
- Network analysis and CPM

Mathematical Programming

Operations research was born with the increasing need to solve optimal resource allocation during WWII.

- Air Battle of Britain
- North Atlantic supply routing problems
- Optimal allocation of military convoys in Europe

Dantzig (1947) is credited with the first solutions to linear programming problems using the Simplex Method

Resource Allocation

Linear Programming Applications

- Allocation of products in the market
- Mixing problems
- Allocation of mobile resources in infrastructure construction (e.g., trucks, loaders, etc.)
- Crew scheduling problems
- Network flow models
- Pollution control and removal
- Estimation techniques

Linear Programming

General Formulation

Maximize $\sum_{j=1}^{n} c_j x_j$

subject to: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for $i = 1, 2, \ldots, m$

$x_j \geq 0$ for $j = 1, 2, \ldots, n$

Linear Programming

Maximize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$$

and $x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0$

Linear Programming

$$\sum_{j=1}^{n} c_j x_j$$ Objective Function (OF)

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$ Functional Constraints (*m* of them)

$x_j \geq 0$ Nonnegativity Conditions (*n* of these)

x_j are decision variables to be optimized (min or max)

c_j are costs associated with each decision variable

Linear Programming

a_{ij} are the coefficients of the functional constraints

b_i are the amounts of the resources available (RHS)

Some definitions

Feasible Solution (FS) - A solution that satisfies all functional constraints of the problem

Basic Feasible Solution (BFS)- A solution that needs to be further investigated to determine if optimal

Initial Basic Feasible Solution - a BFS used as starting point to solve the problem

LP Example (Construction)

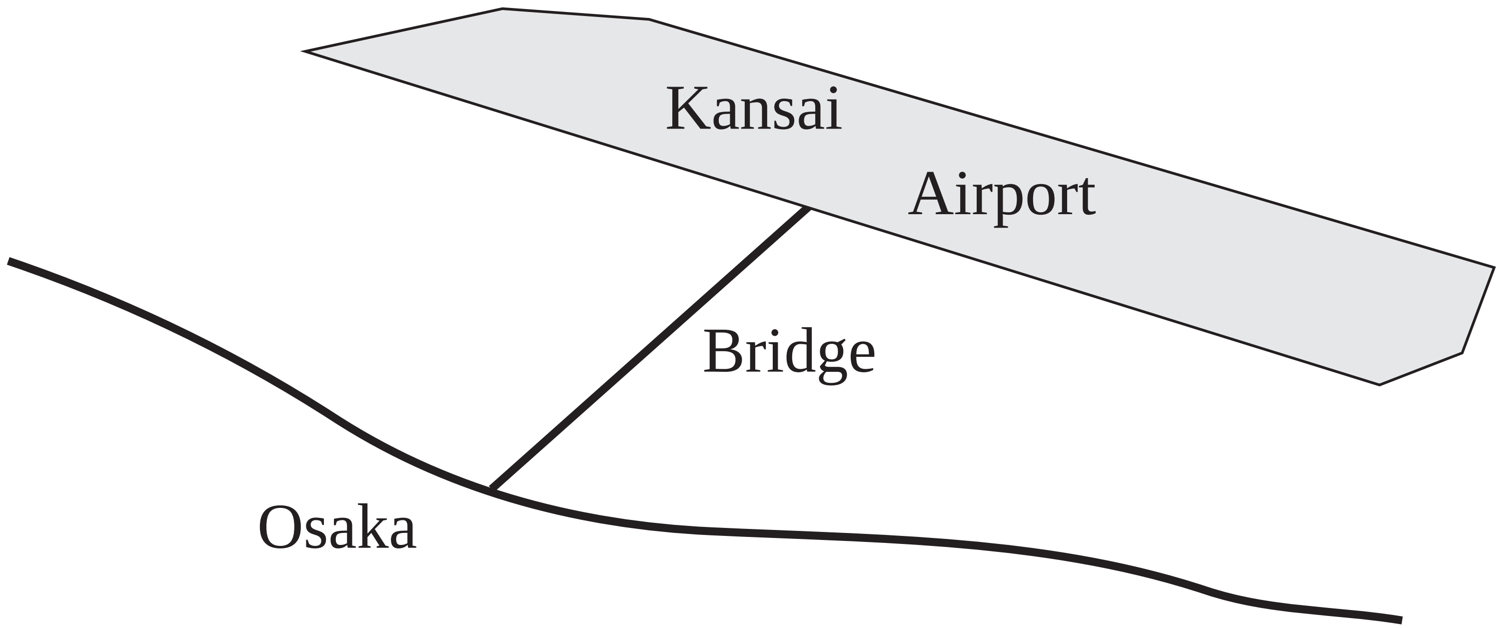
During the construction of an off-shore airport in Japan the main contractor used two types of cargo barges to transport materials from a fill collection site to the artificial island built to accommodate the airport.

The types of cargo vessels have different cargo capacities and crew member requirements as shown in the table:

Vessel Type	Capacity (m-ton)	Crew required	Number available
Fuji	300	3	40
Haneda	500	2	60

Osaka Bay Model

According to company records there are 180 crew members in the payroll and all crew members are trained to either manage the “Haneda” or “Fuji” vessels.



Osaka Bay Model

Mathematical Formulation

Maximize $Z = 300x_1 + 500x_2$

subject to: $3x_1 + 2x_2 \leq 180$

$x_1 \leq 40$

$x_2 \leq 60$

$x_1 \geq 0$ and $x_2 \geq 0$

Note: let x_1 and x_2 be the no. “Fuji” and “Haneda” vessels

Osaka Bay LP Model

Maximize $Z = 300x_1 + 500x_2$

Solution:

a) Covert the problem in standard (canonical) form

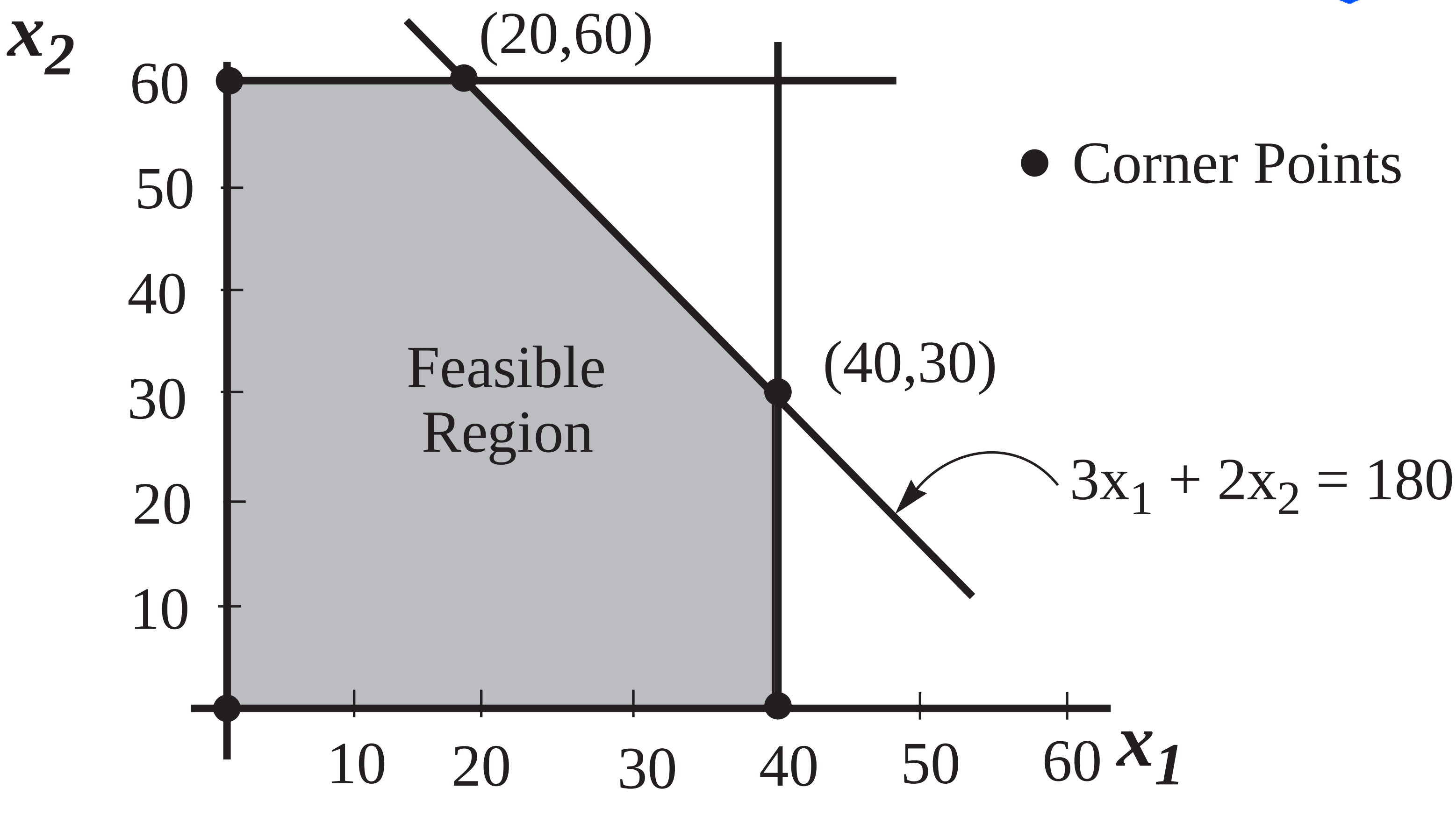
subject to: $3x_1 + 2x_2 + x_3 = 180$

$$x_1 + x_4 = 40$$

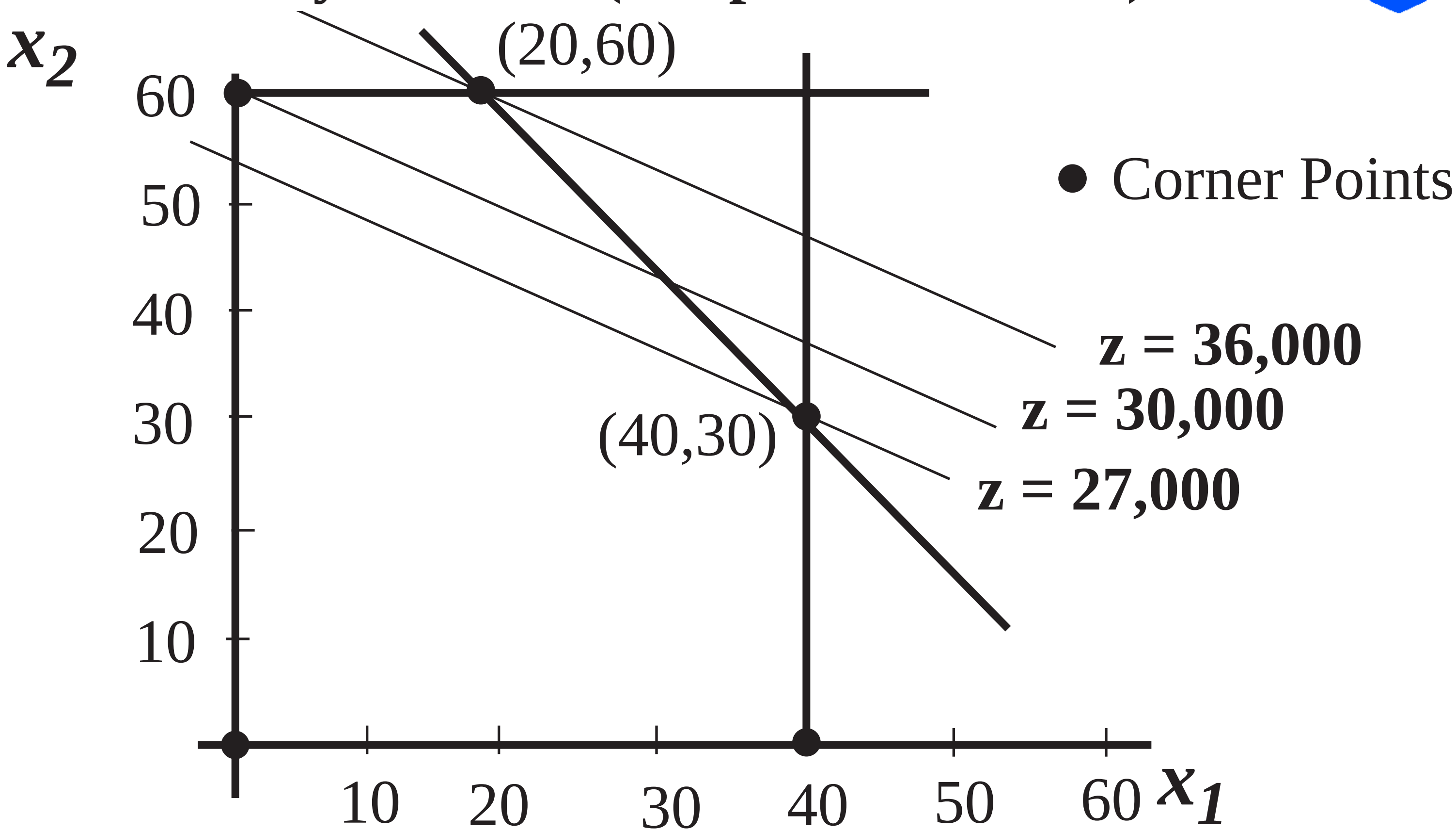
$$x_2 + x_5 = 60$$

$x_1 \geq 0$ and $x_2 \geq 0$

Osaka Bay Problem (Graphical Solution)



Osaka Bay Problem (Graphical Solution)



Note: Optimal Solution (x_1, x_2) = (20,60) vessels

Osaka Bay Problem (Simplex)

Arrange objective function in standard form to perform Simplex tableaus

$$Z - 300x_1 - 500x_2 = 0$$

$$3x_1 + 2x_2 + x_3 = 180$$

$$x_1 + x_4 = 40$$

$$x_2 + x_5 = 60$$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Note: x_3, x_4, x_5 are **slack variables**

Osaka Bay Example (Initial Tableau)

BV	**z**	x_1	x_2	x_3	x_4	x_5	**RHS**
z	**1**	**-300**	**-500**	**0**	**0**	**0**	**0**
x_3	0	3	2	1	0	0	180
x_4	0	1	0	0	1	0	40
x_5	0	0	1	0	0	1	60

BV = x_3, x_4, x_5 and NBV = x_1, x_2

Solution: (x_1, x_2,x_3, x_4, x_5) = (0,0,180,40,60)

Osaka Bay Example (Initial Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
z	**1**	**-300**	**-500**	**0**	**0**	**0**	**0**	
x_3	0	3	2	1	0	0	180	90
x_4	0	1	0	0	1	0	40	inf
x_5	0	0	1	0	0	1	60	60

x_2 improves the objective function more than x_1

Leaving BV = x_5 : New BV = x_2

Osaka Bay Example (Second Tableau)

BV	**z**	x_1	x_2	x_3	x_4	x_5	**RHS**	**ratio**
z	**1**	**-300**	**0**	**0**	**0**	**500**	**30,000**	
x_3	0	3	0	1	0	0	60	20
x_4	0	1	0	0	1	0	40	40
x_2	0	0	1	0	0	1	60	inf

x_1 improves the objective function the maximum

Leaving BV = x_3 : New BV = x_1

Osaka Bay Example (Final Tableau)

BV	**z**	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\boldsymbol{x_4}$	$\boldsymbol{x_5}$	**RHS**
z	**1**	**0**	**0**	**100**	**0**	**300**	**36,000**
$\boldsymbol{x_1}$	0	1	0	1/3	0	0	20
$\boldsymbol{x_4}$	0	0	0	-1/3	1	2/3	20
$\boldsymbol{x_2}$	0	0	1	0	0	1	60

Note: All NVB coefficients are positive or zero in tableau

Optimal Solution: $(\boldsymbol{x_1}, \boldsymbol{x_2}, x_3, x_4, x_5) = (\mathbf{20}, \mathbf{60}, 0, 20, 0)$

Solution Using Excel Solver

- Solver is a Generalized Reduced Gradient (GRG2) nonlinear optimization code
- Developed by Leon Lasdon (UT Austin) and Allan Waren (Cleveland State University)
- Optimization in Excel uses the Solver add-in.
- Solver allows for one function to be minimized, maximized, or set equal to a specific value.
- Convergence criteria (convergence), integer constraint criteria (tolerance), and are accessible through the OPTIONS button.

Excel Solver

- Excel can solve simultaneous linear equations using matrix functions
- Excel can solve one nonlinear equation using Goal Seek or Solver
- Excel does not have direct capabilities of solving n multiple nonlinear equations in n unknowns, but sometimes the problem can be rearranged as a minimization function

Osaka Bay Problem in Excel

Optimization Problem for Osaka Bay

Decision Variables		
x1	20	Number of Ships Type 1
x2	60	Number of Ships Type 2

Objective Function	
300 x1 + 500 x2	36000

Objective function
Stuff to be solved

Constraint Equations			
	Formula		
3 x1 + 2 x2 <= 180	180	<=	180
x1 <= 40	20	<=	40
x2 <= 60	60	<=	60
x1 >= 0	20	>=	0
x2 >= 0	60	>=	0

Osaka Bay Problem in Excel

Optimization Problem for Osaka Bay

Decision variables (what your control)

Decision Variables			
x1	20		Number of Ships Type 1
x2	60		Number of Ships Type 2

Objective Function	
300 x1 + 500 x2	36000

Constraint Equations			
	Formula		
3 x1 + 2 x2 <= 180	180	<=	180
x1 <= 40	20	<=	40
x2 <= 60	60	<=	60
x1 >= 0	20	>=	0
x2 >= 0	60	>=	0

Osaka Bay Problem in Excel

Optimization Problem for Osaka Bay

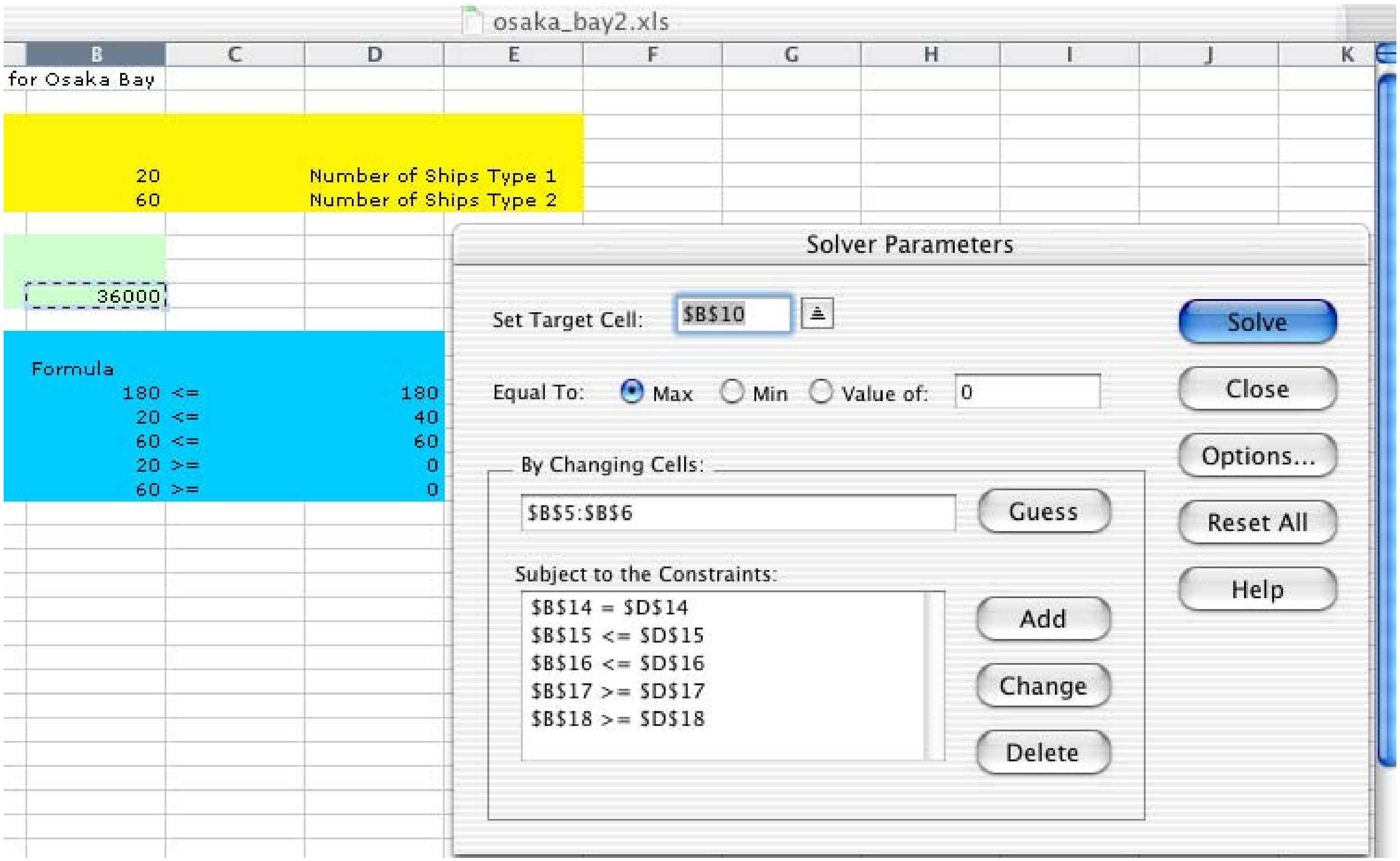
Decision Variables		
x1	20	Number of Ships Type 1
x2	60	Number of Ships Type 2

Objective Function	
300 x1 + 500 x2	36000

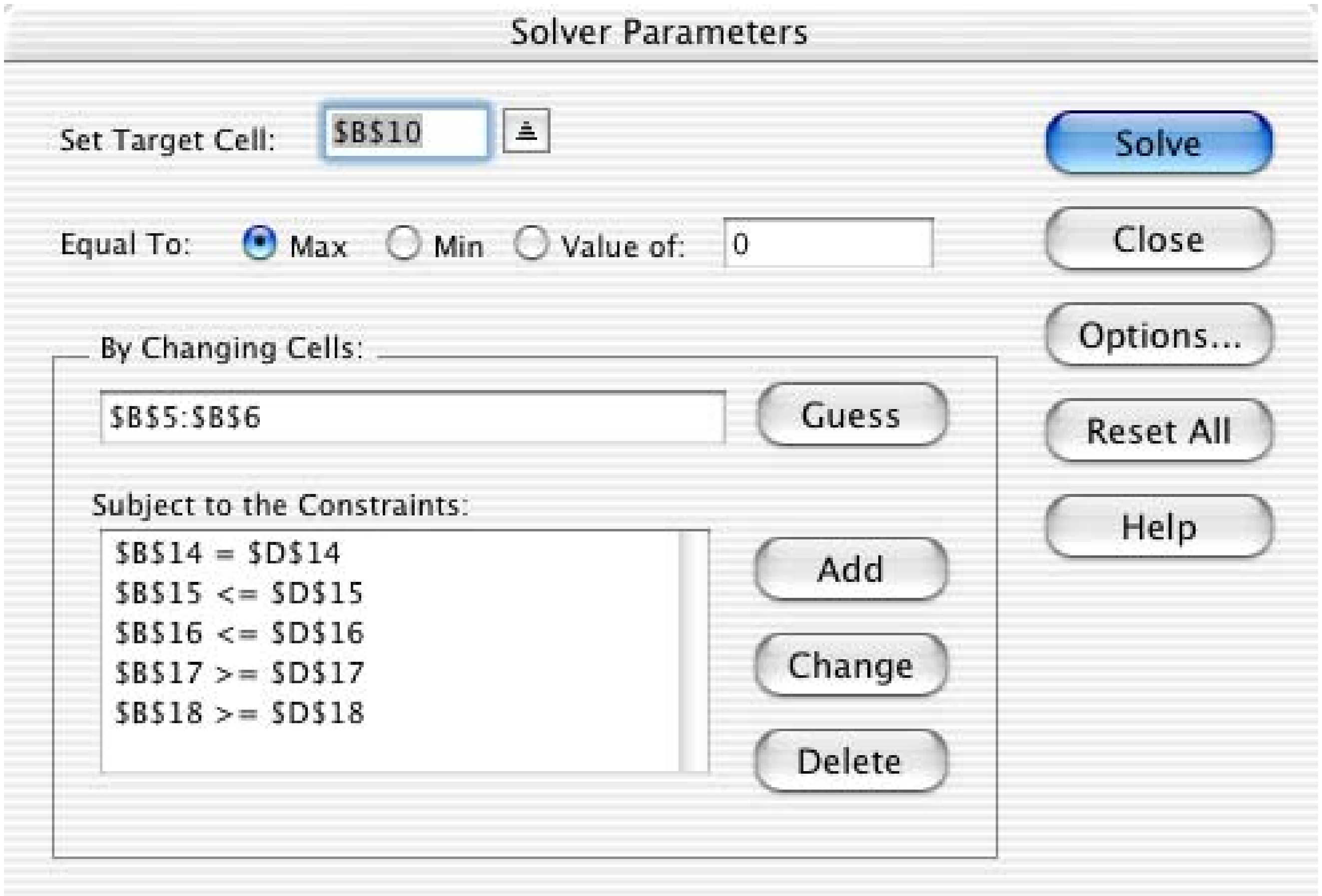
Constraint equations (limits to the problem)

Constraint Equations			
	Formula		
3 x1 + 2 x2 <= 180	180	<=	180
x1 <= 40	20	<=	40
x2 <= 60	60	<=	60
x1 >= 0	20	>=	0
x2 >= 0	60	>=	0

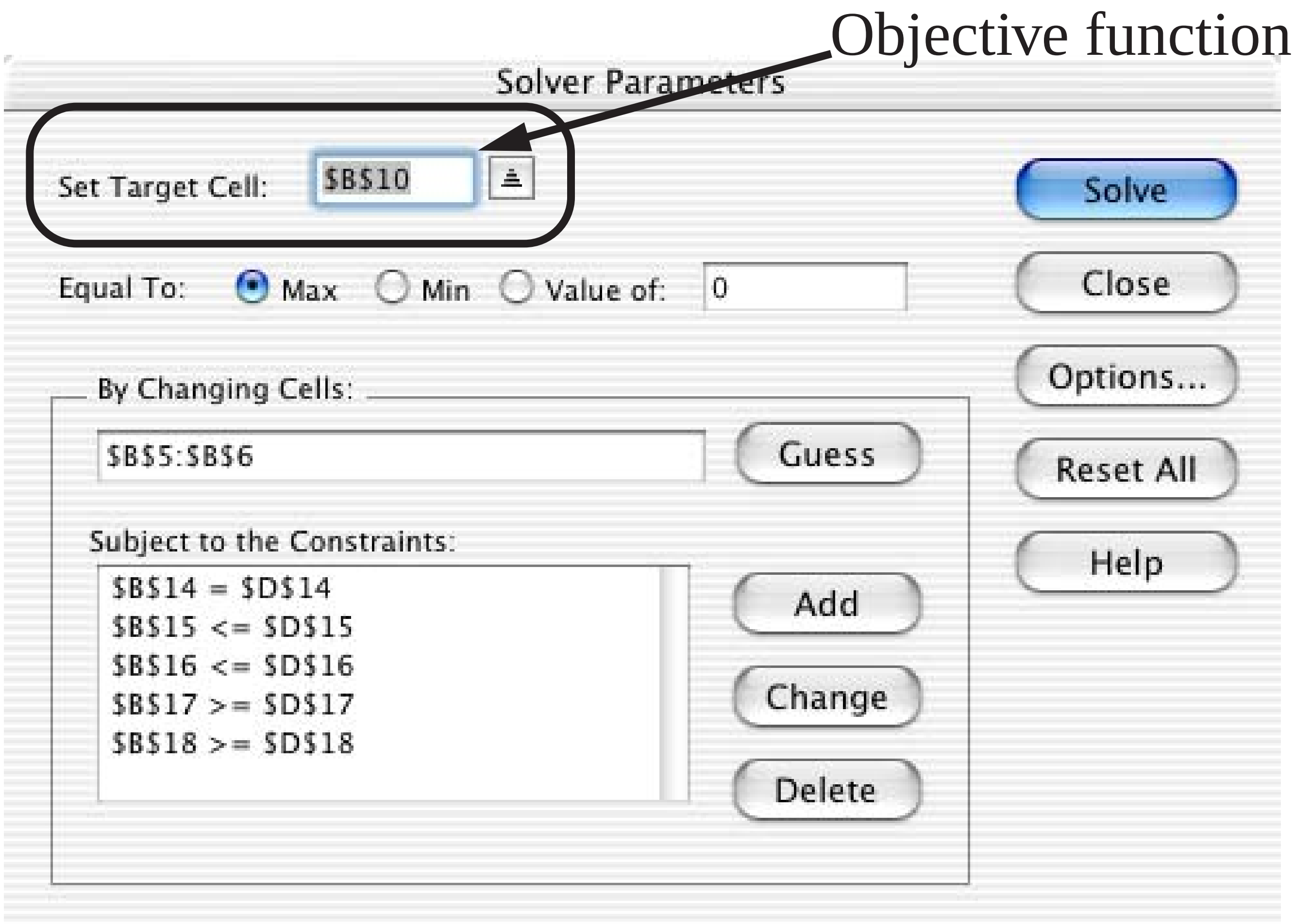
Solver Panel in Excel



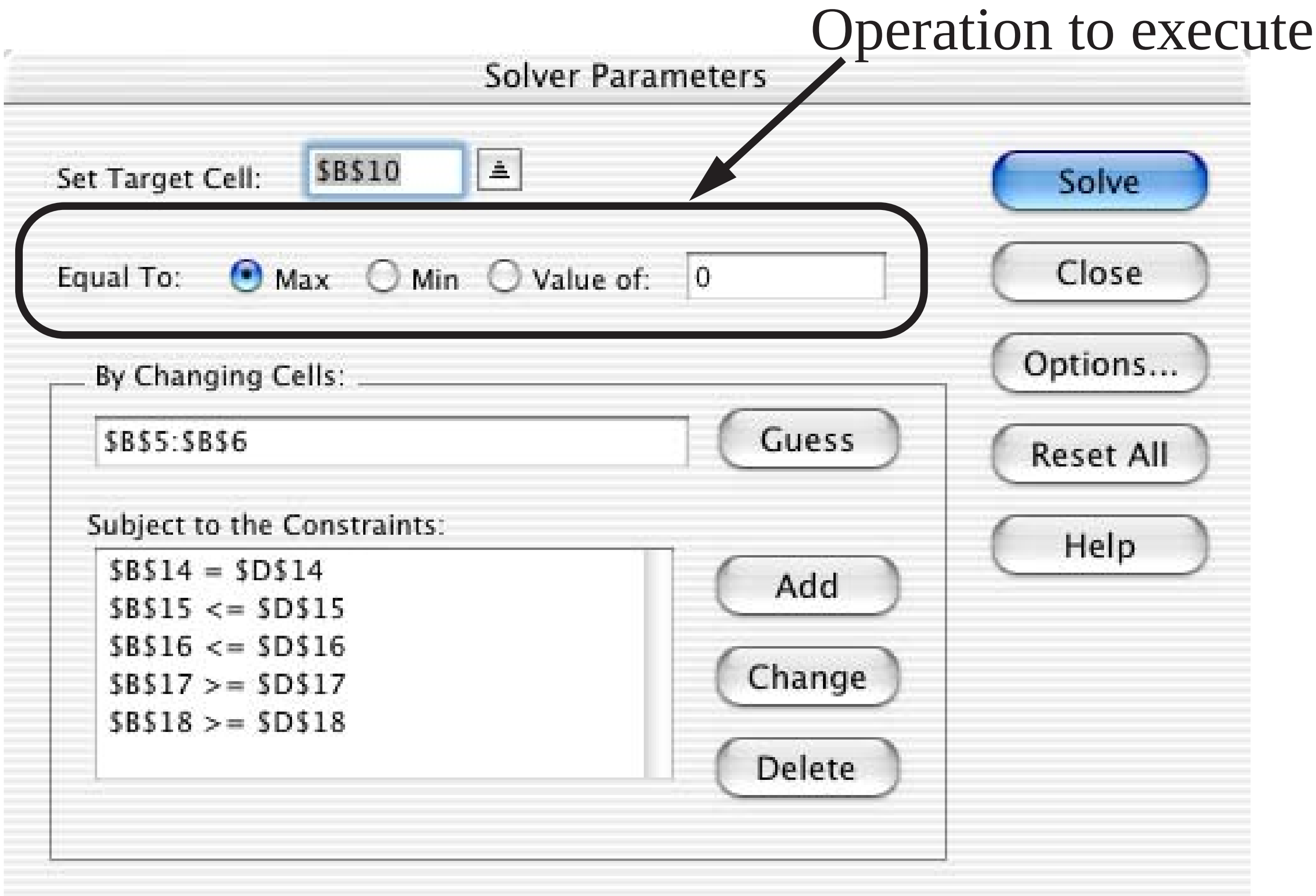
Solver Panel in Excel



Solver Panel in Excel

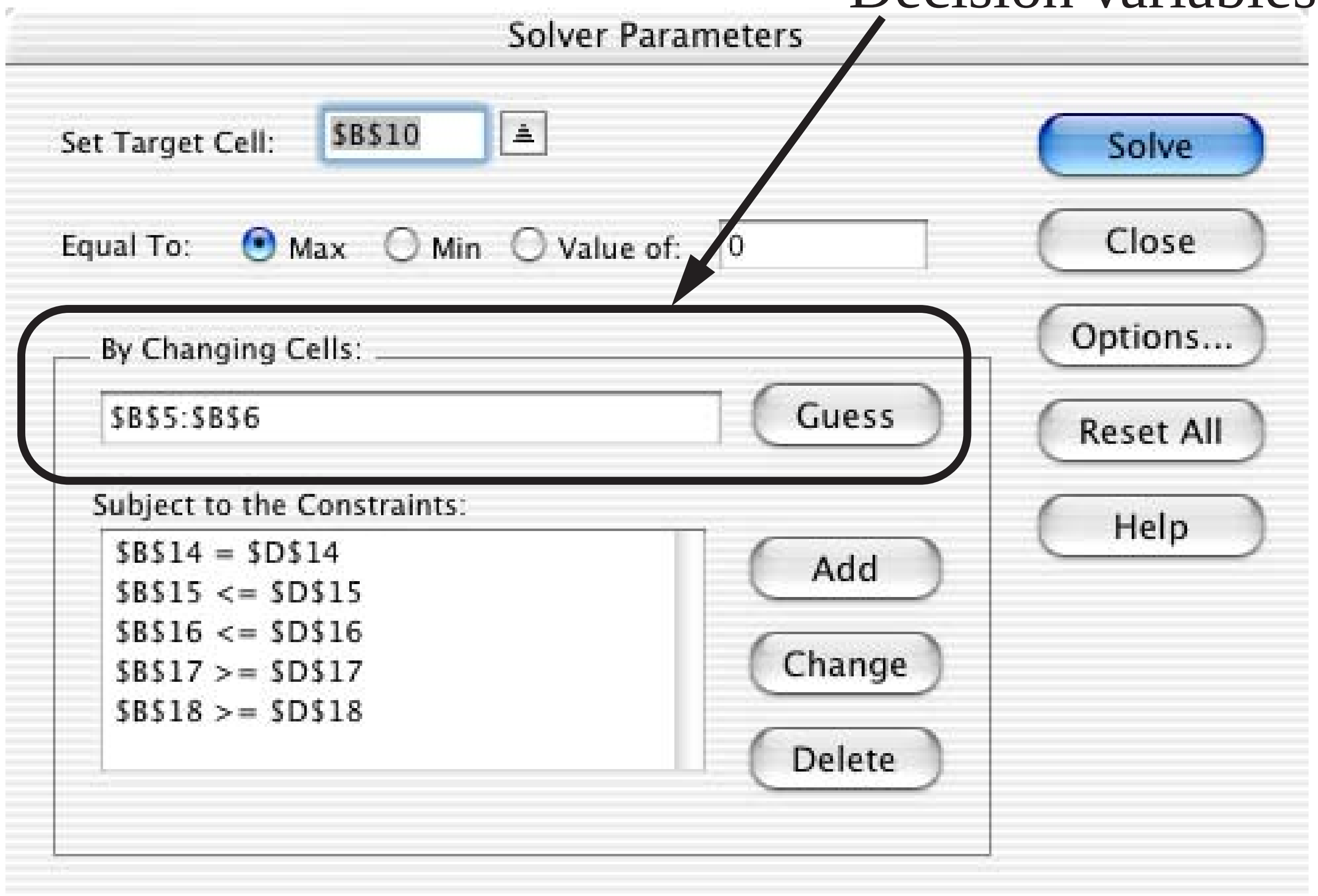


Solver Panel in Excel



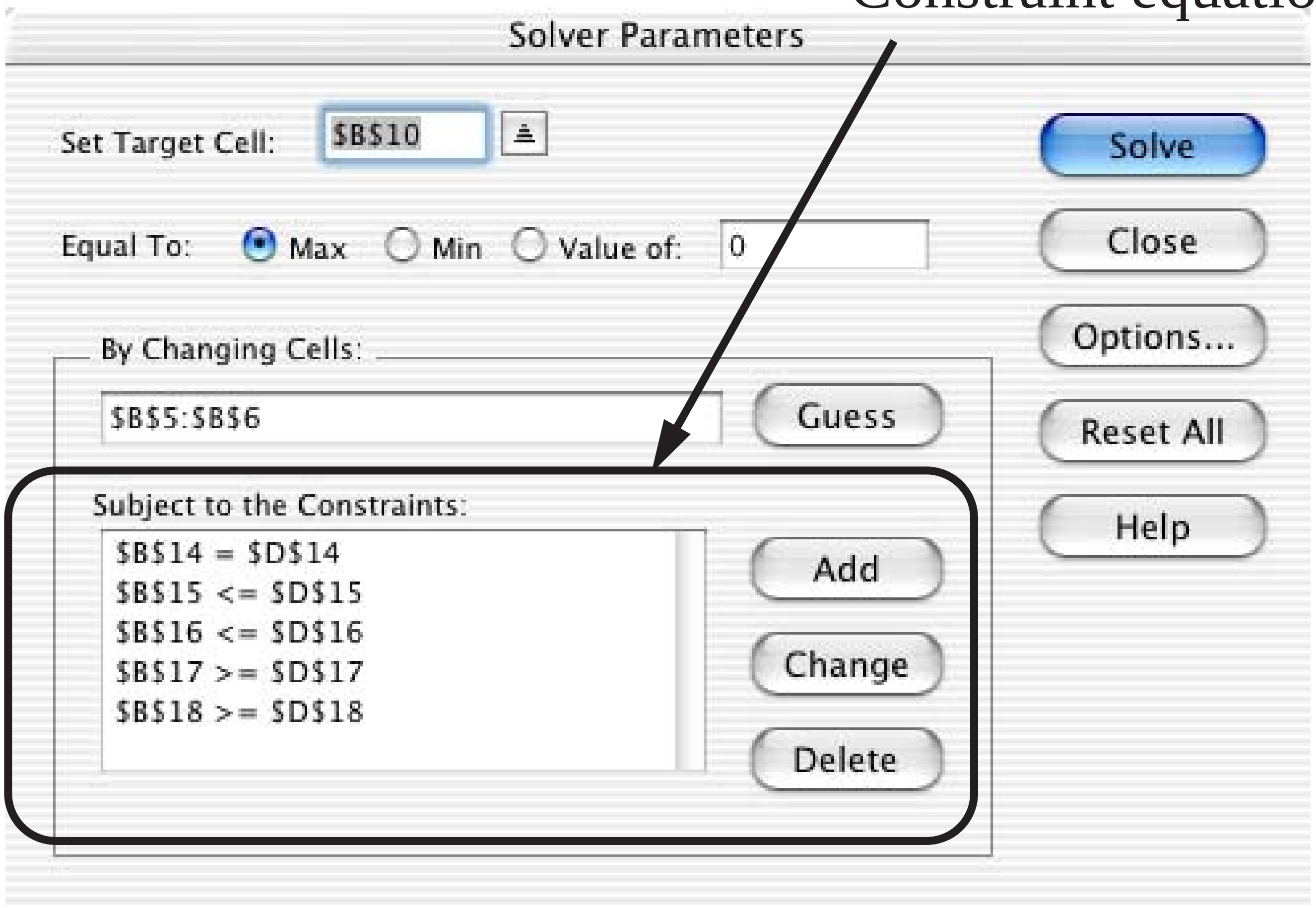
Solver Panel in Excel

Decision variables

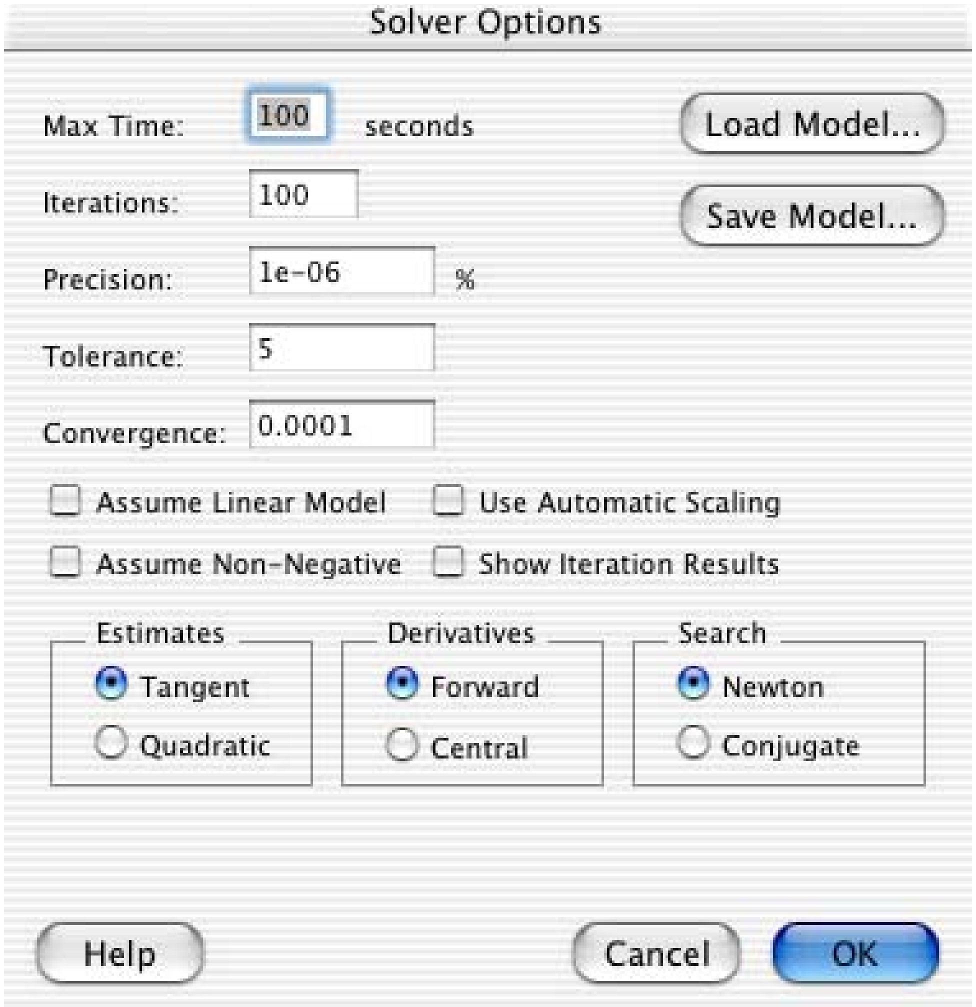


Solver Panel in Excel

Constraint equations



Solver Options Panel Excel



Excel Solver Limits Report

- Provides information about the limits of decision variables

osaka_bay2.xls

Microsoft Excel 10.1 Limits Report
Worksheet: [Workbook1]Sheet1
Report Created: 3/10/2003 5:04:26 AM

Cell	Target Name	Value
B10	300 x1 + 500 x2	36000

Cell	Adjustable Name	Value	Lower Limit	Target Result	Upper Limit	Target Result
B5	x1	20	0	30000	20	36000
B6	x2	60	1.33227E-15	6000	60	36000

Excel Solver Sensitivity Report

- Provides information about shadow prices of decision variables

Microsoft Excel 10.1 Sensitivity Report
Worksheet: [osaka_bay2.xls]Sheet1
Report Created: 3/10/2003 5:47:49 AM

Adjustable Cells

Cell	Name	Final Value	Reduced Gradient
B5	x1	20	0
B6	x2	60	0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
B18	x2 >= 0 Formula	60	0
B15	x1 <= 40 Formula	20	0
B16	x2 <= 60 Formula	60	300
B17	x1 >= 0 Formula	20	0
B14	3 x1 + 2 x2 <= 180 Formula	180	100

Osaka Bay Model (Revised)

Mathematical Formulation

Maximize $Z = 300x_1 + 500x_2$

subject to: $3x_1 + 2x_2 = 180$ Revised Constraint

$x_1 \leq 40$

$x_2 \leq 60$

$x_1 \geq 0$ and $x_2 \geq 0$

Note: let x_1 and x_2 be the no. “Fuji” and “Haneda” vessels

Osaka Bay Model (Revised)

Maximize $Z = 300x_1 + 500x_2$

a) Covert the problem in standard form

subject to: $3x_1 + 2x_2 = 180$

$x_1 + x_3 = 40$

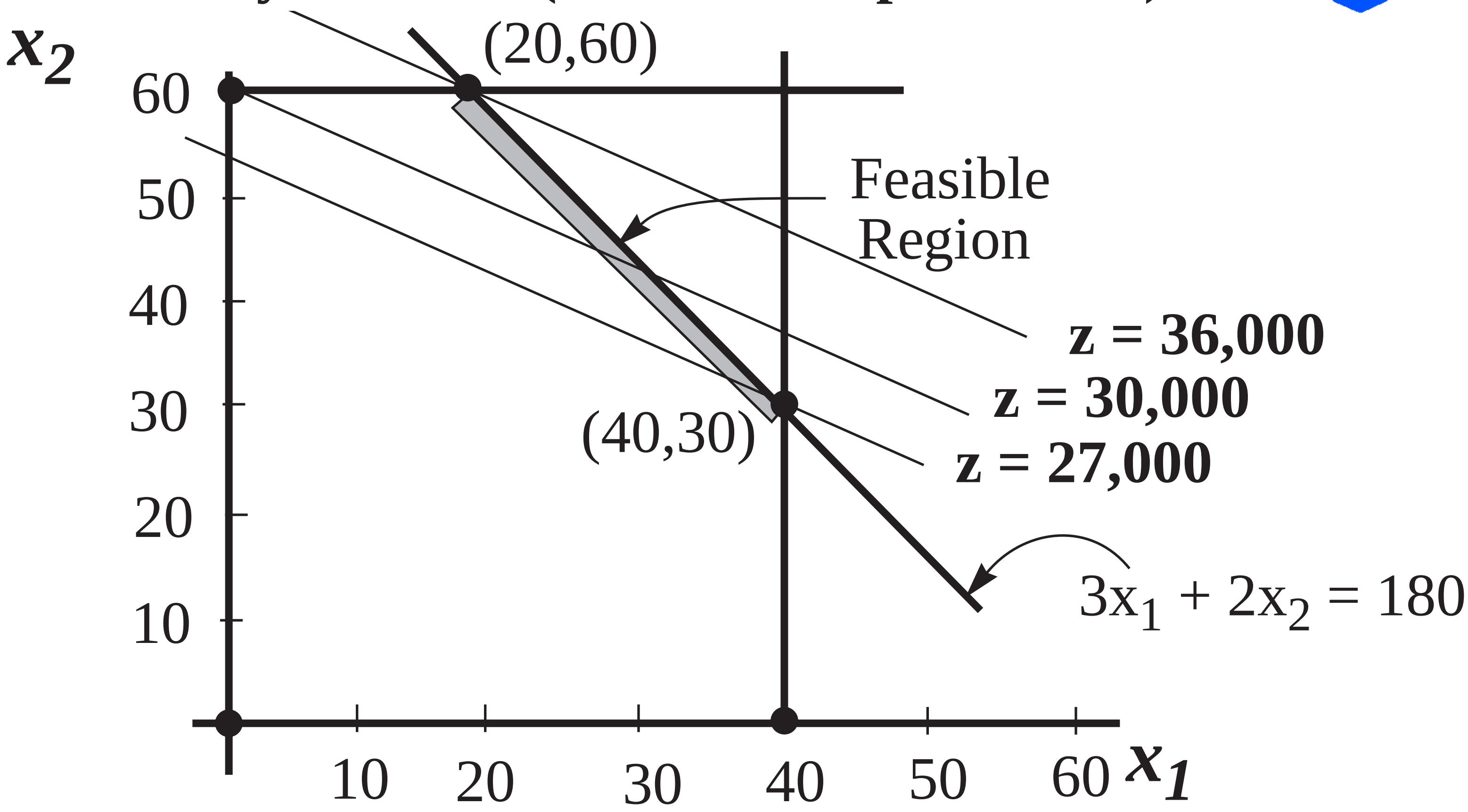
$x_2 + x_4 = 60$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and $x_4 \geq 0$

- Note: Problem lacks an intuitive IBFS (see first constraint)

- Note that setting $x_1 = 0$ and $x_2 = 0$ produces finite integer values for x_3 and x_4 (40 and 60, respectively) but fails to provide and adequate solution for constraint (1).
- This requires a reformulation step where another variable is added to the problem to identify an IBFS
- Add an artificial variable to the first constraint to solve the problem
- Adding an artificial variable in the constraint equation requires the addition of a large penalty to the objective function (z) to avoid this artificial variable being part of the solution

Osaka Bay Problem (Revised Graphical Sol.)



Osaka Bay Model (Revised)

Maximize $Z = 300x_1 + 500x_2$

a) Add an artificial variable to the initial “equal to” constraint

subject to: $3x_1 + 2x_2 + \bar{x}_5 = 180$

$x_1 + x_3 = 40$

$x_2 + x_4 = 60$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

IBFS is now evident with x_1 and x_2 being zero (NVB).

Revised Solution (Big-M Method)

Revise the **objective function** to drive artificial variable to zero in the optimal solution. M is a large positive number.

Maximize $Z = 300x_1 + 500x_2 - Mx_5$

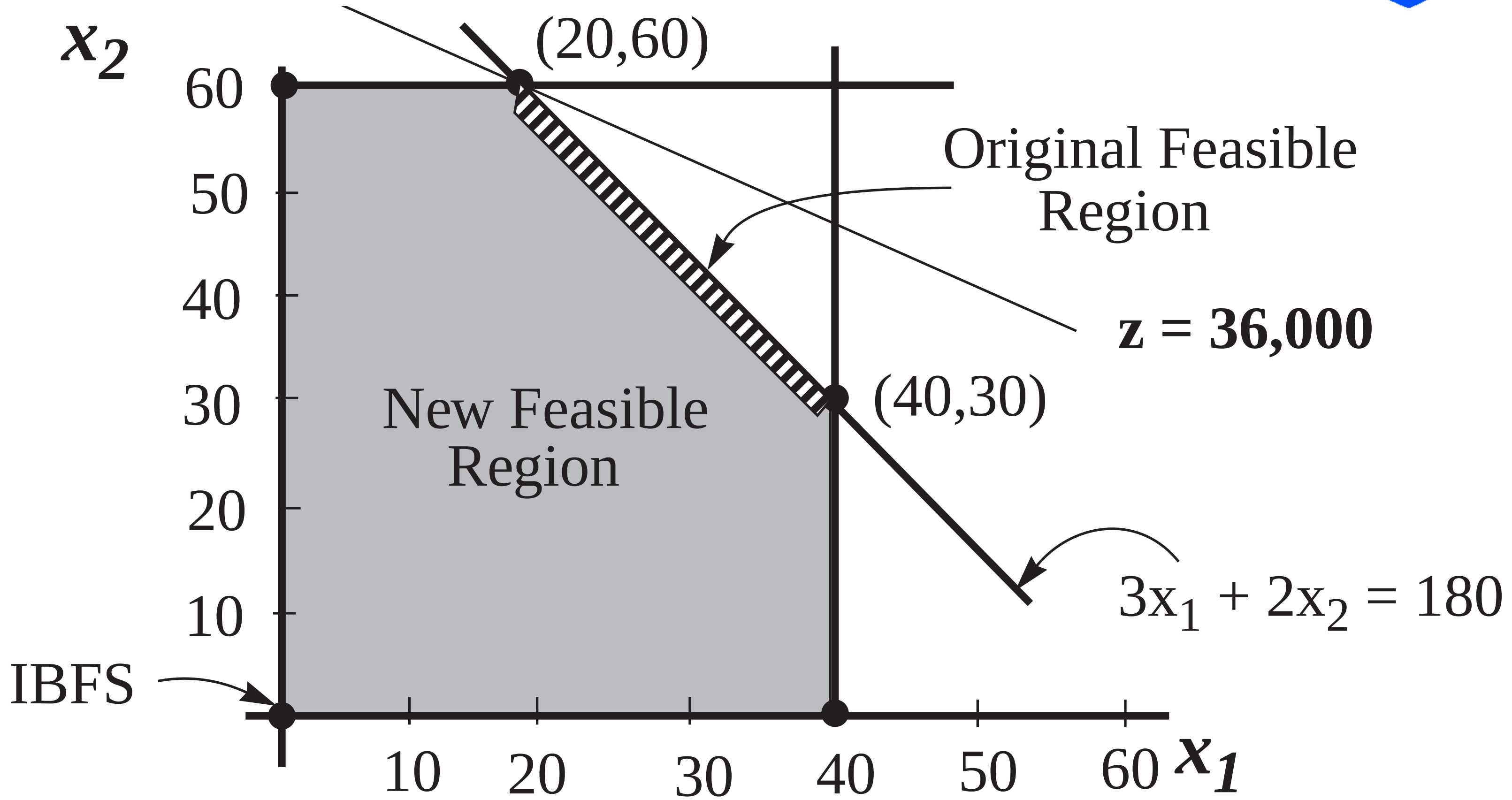
subject to: $3x_1 + 2x_2 + x_5 = 180$

$x_1 + x_3 = 40$

$x_2 + x_4 = 60$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Osaka Bay LP (Expanded Feasible Region)



Revised Solution (Big-M Method)

Rearrange the OF and constraints before solving

Maximize $Z - 300x_1 - 500x_2 + Mx_5 = 0$

subject to: $x_1 + x_3 = 40$

$$x_2 + x_4 = 60$$

$$3x_1 + 2x_2 + \bar{x}_5 = 180$$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Note: the “Big M” (or a large penalty) is added to each artificial variable in OF. x_3 and x_4 are slack variables, x_5 is an artificial variable.

Revised Osaka Bay LP (Initial Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**-300**	**-500**	**0**	**0**	**M**	**0**
x_3	0	1	0	1	0	0	40
x_4	0	0	1	0	1	0	60
x_5	0	3	2	0	0	1	180

BV = x_3, x_4, x_5 and NBV = x_1, x_2

Solution: $(x_1, x_2, x_3, x_4, x_5) = (0,0,40,60,180)$

Revised Osaka Bay LP (Initial Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	
z	**1**	**-3M-300**	**-2M-500**	**0**	**0**	**0**	**-180M**	
x_3	0	1	0	1	0	0	40	40
x_4	0	0	1	0	1	0	60	inf
x_5	0	3	2	0	0	1	180	60

x_1 improves the objective function the maximum

Leaving BV = x_3 : New BV = x_1

Revised Osaka Bay LP (2nd Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	
z	**1**	**0**	**-2M-500**	**3M+300**	**0**	**0**	**-60M+ 12000**	
x_1	0	1	0	1	0	0	40	inf
x_4	0	0	1	0	1	0	60	60
x_5	0	0	2	-3	0	1	60	30

x_2 improves the objective function the maximum. Leaving BV $= x_5$: New BV = x_2

Revised Osaka Bay LP (3rd Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS	
z	**1**	**0**	**0**	**-450**	**M+250**	**0**	**27000**	
x_1	0	1	0	1	0	0	40	40
x_4	0	0	0	3/2	1	-1/2	30	20
x_2	0	0	1	-3/2	0	1/2	30	no

x_3 improves the objective function the maximum. Leaving BV = x_4 : New BV = x_3

Revised Osaka Bay LP (Final Tableau)

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	**1**	**0**	**0**	**0**	**300**	**M+100**	**36000**
x_1	0	1	0	0	-2/3	1/3	20
x_3	0	0	0	1	2/3	-1/3	20
x_2	0	0	1	0	-1/2	1/2	60

Note: All NVB coefficients are positive or zero in tableau

Optimal Solution: $(x_1, x_2, x_3, x_4, x_5) = (\mathbf{20, 60}, 20, 0, 0)$

Simplex Method Anomalies

a) Ties for leaving BV - break without arbitration

b) Ties for entering BV - break without arbitration

c) Zero coefficient of NBV in OF (final tableau) - Implies multiple optimal solutions

d) No leaving BV - implies unbounded solution

Steps in the Simplex Method

I) Initialization Step

- Introduce slack variables
- Select original variables of the problems as part of the NBV
- Select slacks as BV

II) Stopping Rule

- The solution is optimal if every coefficient in the OF is nonnegative

- Coefficients of OF measure the rates of change of the OF as any other variable increases from zero

III) Iterative Step

- Determine the entering NBV (pivot column)
- Determine the leaving BV (from BV set) as the first variable to go to zero without violating constraints
- Perform row operations to make coefficients of BV unity in their respective rows
- Eliminate new BV coefficients (from pivot column) from other equations performing row operations

Linear Programming Strategies Using the Simplex Method

- Identify the problem
- Formulate the problem using LP
- Solve the problem using LP
- Test the model (correlation and sensitivity analysis)
- Establish controls over the model
- Implementation
- Model re-evaluation

LP Formulations

Type of Constraint	**How to handle**
$3x_1 + 2x_2 \le 180$	Add a slack variable
$3x_1 + 2x_2 = 180$	Add an artificial variable Add a penalty to OF (BigM)
$3x_1 + 2x_2 \ge 180$	Add a negative slack and a positive artificial variable

LP (Handling Constraints)

Type of Constraint	**Equivalent Form**
$3x_1 + 2x_2 \leq 180$	$3x_1 + 2x_2 + x_3 = 180$
$3x_1 + 2x_2 = 180$	$3x_1 + 2x_2 + x_3 = 180$ $z = c_1x_1 + c_2x_2 - Mx_3$
$3x_1 + 2x_2 \geq 180$	$3x_1 + 2x_2 - x_3 + x_4 = 180$ $z = c_1x_1 + c_2x_2 - Mx_4$

Note: M is a large positive number

Theory Behind Linear Programming (per Hillier and Lieberman)

General Formulation

Maximize $Z = \sum_{j=1}^{n} c_j x_j$

subject to: $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for $i=1,2, \ldots, m$

$x_j \geq 0$ for $j=1,2,\ldots,n$

General LP Formulation (Matrix Form)

Maximize $Z = \boldsymbol{cx}$

subject to: $\boldsymbol{Ax} = \boldsymbol{b}$

$\boldsymbol{x} \geq \boldsymbol{0}$ where:

$\boldsymbol{c}$ is the vector containing the coefficients of the O.F.,

$\boldsymbol{A}$ is the matrix containing all coefficients of the functional constraints,

$\boldsymbol{b}$ is the column vector for RHS coefficients,

$\boldsymbol{x}$ is the vector of decision variables

note that: $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 \ldots & c_n \end{bmatrix}$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}, \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ and matrix $\boldsymbol{A}$

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ a_{m1} & a_{m2\ldots} & a_{mn} \end{bmatrix}$$

Theory Behind the Simplex Method

Addition of slack variables to the problem yields:

$$\boldsymbol{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ x_{n+m} \end{bmatrix}$$ where $\boldsymbol{x}_s$ is a vector of slack variables (m)

New augmented constraints become,

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \boldsymbol{b} \text{ and } \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} \geq \boldsymbol{0}$$

Note: I is an $m \times m$ identity matrix.

Theory Behind the Simplex Method

Basic Feasible Solution. From the system,

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \boldsymbol{b}$$ n Nonbasic Variables (NBV) from the set,

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix}$$ are set to be equal to zero.

This leaves a set of m equations and m unknowns.

These unknowns correspond to the set of basic variables

Theory Behind the Simplex Method

Let the set of basic variables be called $\boldsymbol{x}_B$ and the matrix containing the coefficients of the functional constraints be called $\boldsymbol{A}$ (basis matrix) so that,

$$\boldsymbol{A}\boldsymbol{x}_B = \boldsymbol{b}$$

$$\boldsymbol{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ x_{Bm} \end{bmatrix}$$

The vector $\boldsymbol{x}_B$ is called vector of basic variables.

Theory Behind the Simplex Method

The idea behind each basic feasible solution in the Simplex Algorithm is to eliminate NBV from the set,

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix}$$

and

$$\bar{A} = \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12}\ldots & \bar{a}_{1m} \\ \bar{a}_{21} & \bar{a}_{22}\ldots & \bar{a}_{2m} \\ \bar{a}_{m1} & \bar{a}_{m2} & \bar{a}_{mm} \end{vmatrix}$$ the basis matrix (a square matrix).

Theory Behind the Simplex Method

From simple matrix algebra (solve for $\boldsymbol{x}_B$) from,

$$\bar{\boldsymbol{A}}\boldsymbol{x}_B = \boldsymbol{b}$$

$$(\bar{\boldsymbol{A}})^{-1}\bar{\boldsymbol{A}}\boldsymbol{x}_B = (\bar{\boldsymbol{A}})^{-1}\boldsymbol{b}$$

$$\boldsymbol{x}_B = (\bar{\boldsymbol{A}})^{-1}\boldsymbol{b}$$

if $\boldsymbol{c}_B$ is the vector of the coefficients of the objective function this brings us to the following value of the objective function:

$$Z = \boldsymbol{c}_B\boldsymbol{x}_B = (\bar{\boldsymbol{A}})^{-1}\boldsymbol{b}$$

Theory Behind the Simplex Method

The original set of equations to start the Simplex Method is,

$$\begin{bmatrix} 1 & -\boldsymbol{c} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{A} & I \end{bmatrix} \begin{bmatrix} Z \\ \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{b} \end{bmatrix}$$

after each iteration in the Simplex Method,

$$\boldsymbol{x}_B = (\boldsymbol{A})^{-1}\boldsymbol{b}$$

and $Z = \boldsymbol{c}_B \boldsymbol{x}_B = (\boldsymbol{A})^{-1}\boldsymbol{b}$

The RHS of the new set of equations becomes,

Theory Behind the Simplex Method

$$\begin{bmatrix} Z \\ \boldsymbol{x}_B \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1} \\ \mathbf{0} & (\bar{A})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_B(\bar{A})^{-1} b \\ (\bar{A})^{-1} b \end{bmatrix}$$

$$\begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1} \\ \mathbf{0} & (\bar{A})^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\boldsymbol{c} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{A} & I \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1}-\boldsymbol{c} & \boldsymbol{c}_B(\bar{A})^{-1} \\ \boldsymbol{o} & (\bar{A})^{-1}\boldsymbol{A} & (\bar{A})^{-1} \end{bmatrix}$$

After any iteration,

$$\begin{bmatrix} 1 & \boldsymbol{c}_B(\bar{A})^{-1}-\boldsymbol{c} & \boldsymbol{c}_B(\bar{A})^{-1} \\ \boldsymbol{o} & (\bar{A})^{-1}\boldsymbol{A} & (\bar{A})^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \boldsymbol{x} \\ \boldsymbol{x}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_B(\bar{A})^{-1} b \\ (\bar{A})^{-1} b \end{bmatrix}$$

In tableau format this becomes,

Theory of the Simplex Method

Iteration	BV	Z	Original Variables	Slack Variables	RHS
0	Z	1	$-\boldsymbol{c}$	$\mathbf{0}$	0
	$\boldsymbol{x}_B$	$\mathbf{0}$	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
Any	Z	1	$\boldsymbol{c}_B(\bar{A})^{-1}\boldsymbol{A}-\boldsymbol{c}$	$\boldsymbol{c}_B(\bar{A})^{-1}$	$\boldsymbol{c}_B(\bar{A})^{-1}b$
	$\boldsymbol{x}_B$	$\mathbf{0}$	$(\bar{A})^{-1}\boldsymbol{A}$	$(\bar{A})^{-1}$	$(\bar{A})^{-1}b$

Numerical Example

To illustrate the use of the revised simplex method consider the Osaka Bay example:

Maximize $Z = 300x_1 + 500x_2$

subject to: $3x_1 + 2x_2 \leq 180$

$x_1 \leq 40$

$x_2 \leq 60$

$x_1 \geq 0$ and $x_2 \geq 0$

Note: let x_1 and x_2 be the no. "Fuji" and "Haneda" vessels

note that: $\boldsymbol{c} = \left[\begin{bmatrix} 300 & 500 \end{bmatrix}\right]$ coefficients of real variables

$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix}$, $\boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and matrix $\boldsymbol{A}$

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Theory Behind the Simplex Method

Addition of slack variables to the problem yields:

$$\boldsymbol{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$ where $\boldsymbol{x}_s$ is a vector of slack variables

Executing the procedure for the Simplex Method

Iteration 0:

$$\boldsymbol{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \; (\bar{A})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix}$$

also known,

$c_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and hence $Z = c_B x_B = (A)^{-1} b$ or

$$Z = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix} = 0$$

Iteration 1: (refer to 2nd tableau in Simplex)

Note: substitute values for $\bar{A}$ using columns for x_3, x_4 and x_2 in the original A matrix.

$$\boldsymbol{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_2 \end{bmatrix}, \bar{\boldsymbol{A}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{\boldsymbol{A}}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ and

$$\begin{bmatrix} x_3 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 180 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \\ 60 \end{bmatrix}$$

also known,

$\boldsymbol{c}_B = \begin{bmatrix} 0 & 0 & 500 \end{bmatrix}$ and hence $Z = \boldsymbol{c}_B \boldsymbol{x}_B = (\boldsymbol{A})^{-1} \boldsymbol{b}$ or

$$Z = \begin{bmatrix} 0 & 0 & 500 \end{bmatrix} \begin{bmatrix} 60 \\ 40 \\ 60 \end{bmatrix} = 30000$$

Iteration 2: (refer to 3rd tableau in Simplex)

Note: substitute values for $\bar{A}$ using columns for x_1 , x_4 and x_2 in the original A matrix.

$$\boldsymbol{x}_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix}, \bar{\boldsymbol{A}} = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; \bar{\boldsymbol{A}}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 60 \\ 40 \\ 60 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 60 \end{bmatrix}$$

also known,

$\boldsymbol{c}_B = \begin{bmatrix} 300 & 0 & 500 \end{bmatrix}$ and hence $Z = \boldsymbol{c}_B \boldsymbol{x}_B = (A)^{-1} \boldsymbol{b}$ or

$$Z = \begin{bmatrix} 300 & 0 & 500 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 60 \end{bmatrix} = 36000$$ **Optimal Solution**

Linear Programming Programs

Several computer programs are available to solve LP problems:

- LINDO - Linear INteractive Discrete Optimizer
- GAMS - also solves non linear problems
- MINUS
- Matlab Toolbox - Optimization toolbox (from Mathworks)
- QSB - LP, DP, IP and other routines available (good for students)